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# ASYMPTOTIC MODELLING AND BOUNDARY-LAYER EFFECT FOR FUNCTIONALLY GRADED MICROLAYERED COMPOSITES

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**Abstract.** The aim of contribution is twofold. First, the new asymptotic modelling method for the analysis of functionally graded multilayered two-phase composites (FGM) is formulated. Second, there is proposed an independent (n on asymptotic) modelling procedure which makes it possible to satisfy boundary conditions related to the asymptotic model. For the sake of simplicity considerations are restricted to problems described by scalar elliptic 2-nd order equations, e.g. to the stationary heat conduction problems, but the proposed modelling method can be also applied to investigations of many other problems in thermomechanics of microlayered FGM.

**Keywords:** FGM, mathematical modelling, heat conduction

## OBJECT OF ANALYSIS

Let $\Omega = (0, L) \times \Xi$, $\Xi \subset R^2$, be the region occupied by a composite in the physical space with the Cartesian orthogonal coordinate system $Ox_1x_2x_3$. Subsequently we denote $\partial_k \equiv \partial/\partial x_k$ where subscripts $k$, $l$ run over 1, 2, 3. Let $z = x_1 \in (0, L)$ and divide $(0, L)$ into intervals of length $\lambda = L/m$ where $m > 1$ is a positive integer. Hence, center of intervals are:

$$z_j^\lambda = \frac{\lambda}{2} + (j-1)\lambda, \quad j = 1, \; ..., \; m = \frac{L}{\lambda}$$

Subsequently $\lambda = L/m$ will be treated as a parameter. Moreover, let (·) be the known continuous together with their first derivatives function; $(\cdot) \in C^1([0, L])$ be the known function, such that $(z) \in (0, 1)$ for every $z \in [0, L]$.

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Under aforementioned notation and for an arbitrary parameter λ region Ω can be decomposed into two non-intersecting parts denoted by:

$$\Omega_B^\lambda = \bigcup_{j=1}^{m} (z_j^\lambda - \frac{\lambda}{2}(z_j^\lambda), \quad z_j^\lambda + \frac{\lambda}{2}(z_j^\lambda)) \times \Xi$$

$$\Omega_W^\lambda = \Omega \setminus \Omega_W^\lambda$$

It follows that the system of interfaces between $\Omega_B^\lambda$ and $\Omega_W^\lambda$ is given by:

$$I^\lambda = \Omega \setminus (\Omega_B^\lambda \cup \Omega_W^\lambda)$$

The composite under consideration are assumed to be made of two materials occupied part $\Omega_B^\lambda$ ("black" material) and $\Omega_W^\lambda$ ("white" material), respectively (Fig. 1).

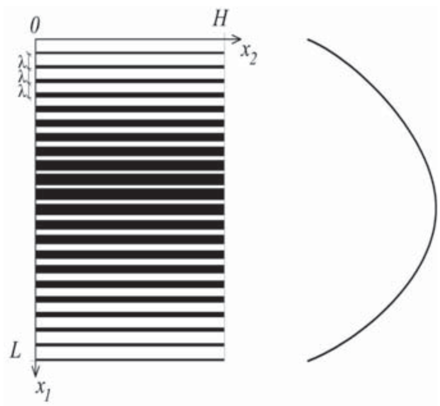


Fig. 1. Scheme of the composite and diagram of mean fraction
Rys. 1. Kompozyt warstwowy z wykresem funkcji nasycenia

Setting ≡, ≡ 1 – we conclude that $_B(z^\lambda{}_j)$, $_B(z^\lambda{}_j)$, $j = 1, \ ..., \ m = \frac{L}{\lambda}$ are mean fractions of both component materials in $(z^\lambda{}_j - \lambda/2, z^\lambda{}_j + \lambda/2)$.

The object of analysis are micro layered composites where parameter λ and function (·) satisfy conditions $\lambda/L << 1$ and $\lambda \left| \frac{d}{dz} \right| << 1$ for every $z \in [0, L]$. If (·) is a constant function then we deal with the well known periodically layered composites. If (·) is not a constant function then the micro layered composite is referred to as "functionally graded". The scheme of this composite is shown in Figure 1 together with diagram of (·).

## AIM OF CONTRIBUTION

The mathematical modelling of boundary value problems in thermomechanics of functionally graded micro layered composites specified in Section 1 will be illustrated on the example of the stationary heat conduction. To this end denote by $A_B$, $A_W$ heat conduction tensor in the "black" and "white" material component respectively. Hence for every $(x_2, x_3) \in \Xi$ we obtain:

$$A_\lambda^{kl}(x_1) = \begin{cases} A_B^{kl}, & \text{if } (x_1, x_2, x_3) \in \Omega_B^\lambda \\ A_W^{kl}, & \text{if } (x_1, x_2, x_3) \in \Omega_W^\lambda \end{cases}$$

where here and subsequently $k$, $l$ run over 1, 2, 3. Under the well known regularity assumptions, we can formulate an elliptic boundary value problem of finding the temperature field $w_\lambda$ in $\Omega$ for equations:

$$q_\lambda^k = -A_\lambda^{kl}\partial_l w_\lambda, \quad \partial_k q_\lambda^k = f \tag{1}$$

when $f$ is known, cf. [Jikov et al. 1994]. Due the discontinuous across $I^\lambda$ and highly oscillating form of functional coefficients $I^\lambda$ to obtain solutions to same aforementioned boundary-value problem is rather a different task.

The first aim of the contribution is to obtain an approximation $w_\lambda^{(1)}$ of $w_\lambda$ in the form:

$$w_\lambda^{(1)} = u + N_\lambda^k \partial_k u \tag{2}$$

where $u$ is a solution to the boundary value problem for equations

$$q_0^k = -A_0^{kl}\partial_l u, \quad \partial_k q_0^k = f \tag{3}$$

matrix $A_0$ is obtained as $G$-limit of matrix $A_\lambda$ where $\lambda \to 0$. The proposed asymptotic procedure is well known for periodic composites, [Jikov et al. 1994] , where $A_0$ represents the constant homogenized elliptic matrix.

It can be seen that approximation $w_\lambda^{(1)}$ of $w_\lambda$ does not satisfy the prescribed boundary conditions for $u(\cdot)$ on $(0, L) \times \partial\Xi$. To eliminate this drawback we can introduce the second approximation $w_\lambda^{(2)}$ of $w_\lambda$ in the form:

$$w_\lambda^{(2)} = u + N_\lambda^k \partial_k u + d_\lambda \tag{4}$$

where $d_\lambda$ satisfies in $\Omega$ equation

$$p_\lambda^k = A_\lambda^{kl}\partial_l d_\lambda, \quad \partial_k p_\lambda^k = 0$$

and boundary condition $d_\lambda = -N_\lambda^{kl}\partial_k u$ on $(0, L) \times \partial\Xi$. A certain simplified method of finding $d_\lambda$, which describes what is called the boundary layer effect, closes this contribution.

## ASYMPTOTIC MODELLING

Let $h_\lambda$ be a continuous bounded functions defined in <0, $L$>; $h_\lambda \in C^0([0, L])$ satisfy conditions:

$$h_\lambda(z_j^\lambda \pm \frac{1}{2}\lambda()) = \pm\frac{\lambda}{2}, \quad j = 1, \ldots, m = \frac{L}{\lambda}$$

and be linear in intervals:

$$(0,\ z_1^\lambda - \frac{1}{2}\lambda(z_1^\lambda)), \quad (z_1^\lambda - \frac{1}{2}\lambda(z_1^\lambda), \quad z_1^\lambda + \frac{1}{2}\lambda(z_1^\lambda)),$$

$$(z_1^\lambda + \frac{1}{2}\lambda(z_1^\lambda), \quad z_2^\lambda - \frac{1}{2}\lambda(z_2^\lambda)), \quad (z_2^\lambda - \frac{1}{2}\lambda(z_2^\lambda), \quad z_2^\lambda + \frac{1}{2}\lambda(z_2^\lambda)), \ldots,$$

$$(z_m^\lambda - \frac{1}{2}\lambda(z_m^\lambda), \quad z_m^\lambda + \frac{1}{2}\lambda(z_m^\lambda)), \quad (z_m^\lambda + \frac{1}{2}\lambda(z_m^\lambda),\ L).$$

The proposed asymptotic modelling is based on assumption that for sufficiently small $\lambda$, $\lambda << L$ an approximation $w_\lambda^{(1)}$ of $w_\lambda$ can be postulated in the form:

$$w_\lambda^{(1)}(x) = u(x) + h_\lambda(x_1)\bar{u}(x)$$

where $x = (x_1, x_2, x_3)$ and functions $u, \bar{u} \in C^1(\Omega)$ are independent on $\lambda$.

The continuity of $A_\lambda^{1k}\partial_k w_\lambda^{(1)}$ on interfaces $I^\lambda$ together with the limit passage $I^\lambda \to I^0$ if $\lambda \to 0$ leads to important assertion that $N_\lambda^k$ in formula (2) is given by:

$$N_\lambda^k = -h_\lambda \frac{\varphi_B\varphi_W(A_B^{1k} - A_W^{1k})}{\varphi_W A_B^{11} - \varphi_B A_W^{11}} \tag{5}$$

and

$$\bar{u} = \frac{\varphi_B\varphi_W(A_B^{1k} - A_W^{1k})}{\varphi_W A_B^{11} - \varphi_B A_W^{11}}\partial_k u$$

The second important assertion is that if $\partial_k q_\lambda^k$ tends weakly to $\partial_k q_0^k$ in $L^2(\Omega)$ under limit passage $\lambda \to 0$ than:

$$q_0^k(x) = \lim_{\lambda\to 0}\frac{1}{\lambda}\int_{x_1-\lambda/2}^{x_1+\lambda/2} q_\lambda^k(z, x_2, x_3)dz$$

Thus we arrive at the final assertion that for $\lambda \to 0$ equations (1) tend to equations $(2)_1$, where under notation $[A^{1k}] \equiv A_B^{1k} - A_W^{1k}$ we obtain:

$$A_0^{kl} = \varphi_B A_B^{lk} + \varphi_W A_W^{kl} - \frac{\varphi_B\varphi_W[A^{1k}][A^{1l}]}{\varphi_B A_W^{11} + \varphi_W A_B^{11}} \tag{6}$$

At the same time it can be shown that matrix $A_0$ is $G$-limit of matrix $A_\lambda$ where $\lambda \to 0$. It mean that $A_0$ is uniquely defined and the postulated a priori form of $w_\lambda^{(1)}$ was correctly stated.

If is a constant function $\in (0,1)$, than results of the asymptotic modelling, described by formulas (2), (3) with denotations (5), (6), coincide with there for the $\lambda$-periodic composites.

**BOUNDARY – LAYER EQUATION**

An approximate solution $d_\lambda = d_\lambda(x_1, x_2, x_3)$, $x_1 \in (0, L)$, $(x_2, x_3) \in \Xi$ to the problem specified at the end of Section 2 will be assumed in the form $d_\lambda = h_\lambda v$, where $v(\cdot, x_2, x_3)$ is slowly-varying smooth function for every $(x_2, x_3) \in \Xi$ [Mathematical modelling... 2010]:

$$v(\cdot, x_2, x_3) \in SV_\delta^\lambda([-\frac{\lambda}{2}, \frac{\lambda}{2}]) \subset C^1([0, L])$$

It means that for every integrable function $\psi \in L(0, L)$ mean values $< \psi v > (x_1)$, $x_1 \in [\frac{\lambda}{2}, L - \frac{\lambda}{2}]$ can be approximated by $< \psi > (x_1) v(x_1, x_2, x_3)$ where $< \psi > (x_1) \equiv \frac{1}{\lambda} \int_{x_1 - \lambda/2}^{x_1 + \lambda/2} \psi(z) dz$.

A similar condition also holds true for all derivatives of $v$.

We recall that $p_\lambda^k = -A_\lambda^{kl} \partial_l (h_\lambda v)$. Instead of condition $\partial_k p_\lambda^k = 0$ shall postulated that $< h_\lambda \partial_k p_\lambda^k >= 0$. Obviously $< \partial_1 (h_\lambda p_\lambda^1) > (x_1) = 0$ for every $x_1 \in I^\lambda$. It will be assumed that this above is neglected for every $x_1 \in (\frac{\lambda}{2}, L - \frac{\lambda}{2})$. For the sake of simplicity let us also assume that $A_\lambda^{kl} = \delta^{kl} \kappa_\lambda$ restricting analysis to composites with isotropic constituents. In this case we arrive at equation:

$$< (h_\lambda)^2 \kappa_\lambda > (x_1) \delta^{\alpha\beta} \partial_\alpha \partial_\beta v - < (\partial_1 h_\lambda)^2 \kappa_\lambda > (x_1) v = 0$$

where $\alpha$, $\beta$ run oper 2,3.

Let $(x_2, x_3) \in (0, L_2) \times (0, L_3)$ and denote $\Gamma = (0, L) \times \{0\} \times (0, L_3)$ as a part of the boundary of region $\Omega$. The boundary layer effect related to $\Gamma$ will be modeled by assumption that derivatives $\partial_2 v$ are sufficiently large when compared to $\partial_1 v$ and $\partial_3 v$. Thus, the pertinent boundary layer equation takes the form:

$$< (h_\lambda)^2 \kappa_\lambda > (x_1) \partial_2^2 v - < (\partial_1 h_\lambda)^2 \kappa_\lambda > (x_1) v = 0 \tag{7}$$

and has to hold in $\left(\frac{\lambda}{2}, L - \frac{\lambda}{2}\right) \times (0, L_2) \times (0, L_3)$. On the boundary $\Gamma = \left(\frac{\lambda}{2}, L - \frac{\lambda}{2}\right) \times \{0\} \times$ $\times (0, L_3)$ function $v(\cdot)$ has to satisfy condition:

$$v = -\frac{\varphi_B \varphi_W [A^{11}]}{\varphi_B A_W^{11} + \varphi_W A_B^{11}} \partial_1 u \tag{8}$$

At the same time values of $v(\cdot)$ on $\left(\frac{\lambda}{2}, L - \frac{\lambda}{2}\right) \times (0, L_2) \times (0, L_3)$ hale to be negligibly small.

For $\Gamma=\left(\frac{\lambda}{2},L-\frac{\lambda}{2}\right)\times\{L\}\times(0,L_3)$ boundary laser equation and pertinent boundary condition are also represented by formulas (7) and (8) respectively. For $\Gamma=\left(\frac{\lambda}{2},L-\frac{\lambda}{2}\right)\times(0,L_2)\times\{L_3\}$ derivatives $\partial_2^2$ in (7) hale to be replaced by $\partial_3^2$. On plans $\{0\}\times\Xi$, $\{L\}\times\Xi$ the boundary layer effect does not exist.

If region $\Xi$ on $Ox_2x_3$ – plane has a smooth boundary $\partial\Xi$ and the smallest curvature radius of $\partial\Xi$ is sufficiently large when compared to $\lambda$ then the boundary laser equation has the form:

$$<(h_\lambda)^2\kappa_\lambda>(x_1)\partial_n^2 v-<(\partial_1 h_\lambda)^2\kappa_\lambda>(x_1)v=0 \tag{9}$$

where $\partial_n$ is a derivatives in the direction normal to the boundary $\partial\Xi$.

Evidently, equation (9) has to hold for every $x_1\in\left(\frac{\lambda}{2},L-\frac{\lambda}{2}\right)$ only in near – boundary part of region $\Xi$. Outside this part function $\Xi$ has to attains absolute values negligibly small when compared to absolute values of the right – hand side of formula (8). More detailed discussion boundary layer equations (9) can be found in [Woźniak 2010].

## CONCLUDING REMARK

The obtained results can be also applied to problems in which $\varphi(\cdot)\in C^1(\bar{\Omega})$ provided that $\lambda|\nabla\varphi(x)|<<1$ holds for every $x=(x_1,x_2,x_3)\in\Omega$, $|\nabla\varphi(x)|$ is the absolute value of vector $\partial_k v$, $k$ = 1, 2, 3. In this case thicknesses of homogeneous material layers measured along $Ox_1$ – axis are slowly varying along, $Ox_2$ – and $Ox_3$ – axis. Moreover, if $v(\cdot)$ is independent of argument $x_1\in(0,L)$ when we deal with functionally gradem composites which are locally periodic i.e. periodic in the $Ox_1$ axis direction for an arbitraży but fixed $(x_2,x_3)\in\Xi$.

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## MODELOWANIE ASYMPTOTYCZNE I EFEKT WARSTWY BRZEGOWEJ DLA WARSTWOWYCH KOMPOZYTÓW GRADIENTOWYCH

**Streszczenie.** W pracy sformułowano dwa cele. Pierwszy z nich to opracowanie asymptotycznej metody modelowania dwufazowych kompozytów warstwowych o strukturze gradientowej. Drugim celem jest wyprowadzenie równania tzw. warstwy brzegowej, które dokładnie spełnia warunki brzegowe postulowane w ramach przybliżenia asymptotycznego. Rozważania przeprowadzono na przykładzie stacjonarnego zagadnienia przewodnictwa cieplnego. Proponowana procedura może być uogólniona na inne zagadnienia termomechaniki kompozytów gradientowych.

**Słowa kluczowe:** FGM, matematyczne modelowanie, przewodnictwo ciepła

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