EFFECTIVE HEAT CONDUCTIVITIES IN CERTAIN BIPERIODICALLY STRATIFIED COMPOSITES

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Abstract. Biperiodically stratified composites are characterized by two kinds of stratification which are realized independently on the micro-scale and mezzo-scale. An example of the above biperiodic structure is shown in Figure 1. The aim of the note is to propose a certain new general homogenization approach leading to effective heat conductivities as well as to illustrate this approach by some specific cases of stratifications.

Key words: heat conduction, biperiodical composite materials, mezzo homogenization, macro homogenization

INTRODUCTION

It is known that the effective properties of periodically stratified composites (e.g. heat conductors) are uniquely determined by the asymptotic homogenization procedure. General foundations of this procedure are detailed in monographs [Bensoussan et al. 1978, Bakhvalov and Panasenko 1984, Jikov et al. 1994] and a number of different applications of the homogenization theory can be found in large number of papers. From the physical point of view these effective properties represent composite structure on the macroscopic scale, i.e. as a certain new homogenized material. On the microscopic scale the composite is observed as made of some different component materials. For periodically stratified heat conductors the known homogenization procedure leading from "micro" to "macro" scale is based on the formal limit passage to zero with the thicknesses of the representative composite layer. The obtained results have a simple analytical form for the effective heat conductivities.

In this paper the object of analysis are biperiodically stratified heat conductors. The components of this macro-periodic structure are two micro-stratified composites shown
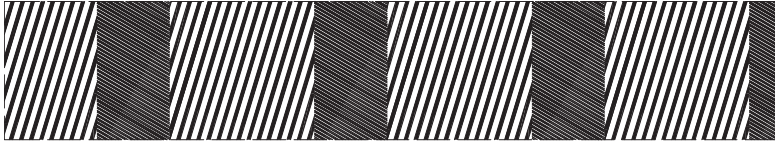
Fig. 1.   A fragment of a cross section of biperiodically stratified heat conductor under considera-
           tion
Rys. 1.   Fragment przekroju przez dwuperiodyczny warstwowy przewodnik ciepła

in Figure 1. Moreover every micro-periodic structure is made of two isotopic components (micro-layers). A fragment of a cross section for this composite is shown in Figure 1.

For the sake of simplicity we shall restrict ourselves to the plane composite structures. In the example shown in Figure 1 the thick black straight line segments represent regions occupied by a certain reinforcement material. At the same time the white part on the scheme represent a matrix material. It can also be seen that we deal here with two kinds of periodic heterogeneity. The first of them will be called the micro-periodic heterogeneity defined as the heterogeneity between the reinforcement material and the matrix material. At the same time we deal here with two micro-periodic structures since there are two directions normal to the interfaces between matrix and reinforcement. From a formal point of view every from two independent micro-structure can be homogenized and hence represented by a new "homogenized" materials. The correspondent procedures will be called the mezzo-homogenization procedures and as the result there is defined as the two-component mezzo-homogenized structure.

Obviously, this new material structure is periodically stratified being the starting point for the second kind of formal homogenization referred to as the macro-homogenization. The constant heat conductivities derived from the macro-homogenization will be referred to as the effective heat conductivities of the biperiodically stratified heat conductor under consideration.

It has to be emphasized that the formal procedure outlined above has a physical sense only if:

1. Periods of heterogeneity for both micro-periodic structures are sufficiently small as compared to the thicknesses of corresponding layers in the mezzo-period structure.

2. The period of a mezzo-periodic structure is sufficiently small as compared to the smallest characteristic length dimension of the region occupied by the biperiodically stratified composite under consideration.

The aim of this consideration is twofold. Firstly to propose the general homogenization procedure for biperiodically stratified composites as heat conductors, which was outlined above. Secondly to illustrate the proposed method by the analysis of some specific cases.

Generally speaking the modelling technique applied in this paper is partly similar to the known reiterated homogenization which from purely analytical point of view was applied in [Avellaneda 1987, Allaire 1992, Allaire and Briane 1996, Braides and Lukkassen 2000, Lions et al. 2000].

We shall also deal whit a concept of multiperiodic structures c.f. [Woźniak Cz. 2002, Woźniak M. and Woźniak Cz. 2003, Jędrysiak and Woźniak Cz. 2006] which has a slightly different meaning then in references mentioned above.

FOUNDATIONS

Mezzo-periodically stratified space

Let $Ox^1x^2x^3$ be Cartesian orthogonal coordinate system in a certain 3-space, where Ox^2-axis is situated on the interface between adjacent macro-layer. This space will be interpreted here as a material composite space. To make subsequent considerations as simple as possible let us interpretative this space as space occupied by two materials which will be referred to as material "A" and material "B". Both materials occupy two disjointed domains Ω_A, Ω_B, which are periodically distributed in two systems of separate layers having thicknesses λ_A, λ_B. The fragment of a mezzo-periodically homogenized space is shown in Figure 2.
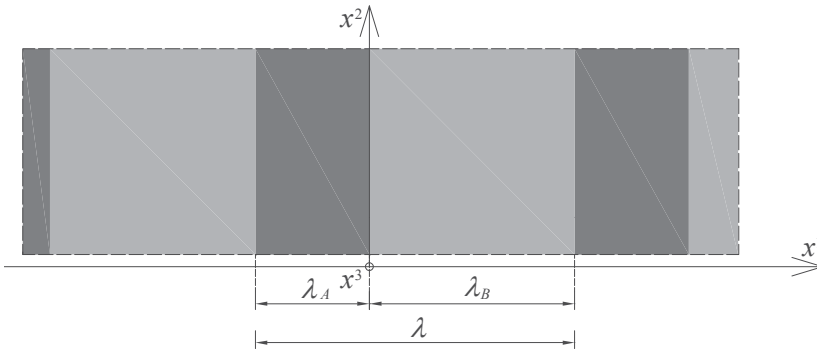

Fig. 2.   A scheme of mezzo-periodically homogenized space
Rys. 2.   Schemat mezzo-periodycznie shomogenizowanej przestrzeni

Setting $m = 0 \pm 1, \; \pm 2, \; ...$ we shall assume that:

$$\Omega_A \equiv \bigcup_m (m\lambda - \lambda_A, m\lambda) \times \mathrm{R}^2$$
$$\Omega_B \equiv \bigcup_m (m\lambda, m\lambda + \lambda_B) \times \mathrm{R}^2 \tag{1}$$

The parts of the space separating domains Ω_A and Ω_B will be denoted by Γ:

$$\Gamma \equiv \bar{\Omega}_A \cap \bar{\Omega}_B \tag{2}$$

The connected parts of Γ will be referred to as the mezzo-interfaces in the whole 3-space.

Micro-periodically stratified layers

So far, it was assumed that the material space under consideration is made of two material components which were referred to as material "A" and material "B", without the specification. Moreover the space is periodically stratified and has the representative layer of a thickness λ composed of two sublayers with thicknesses λ_A and λ_B, $\lambda \equiv \lambda_A + \lambda_B$ measured along Ox^1-axis , c.f. Figure 3.
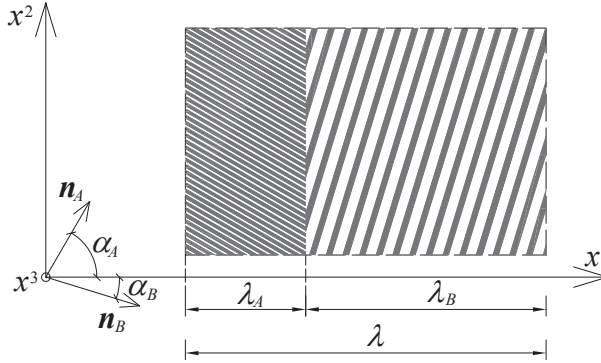

Fig. 3.   A fragment of an representative layer
Rys. 3.   Fragment warstwy reprezentatywnej

Subsequently we shall denote V, $V \in \{A,\, B\}$ restricted considerations to certain layer termed as a composite material "V".

Setting $\alpha_V \in \left[-\dfrac{\pi}{2},\ \dfrac{\pi}{2}\right]$ we shall introduce the orthogonal Cartesian coordinate systems $Ox_V^1 x_V^2 x_V^3$, $V \in \{A, B\}$ defined by:

$$x_V^1 = x^1 \cos \alpha_V + x^2 \sin \alpha_V$$
$$x_V^2 = -x^1 \sin \alpha_V + x^2 \cos \alpha_V \qquad\qquad (3)$$
$$x_V^3 = x^3$$

where Ox_V^1 – axes are orthogonal to micro-stratification.

We have to keep in mind that $\left(x_V^1, x_V^2, x_V^3\right) \in \Omega_V$, $V \in \{A, B\}$ .We also introduce unit vectors in the directions normal to the stratification of materials "A" and "B":

$$\mathrm{n}_V = \left[\cos \alpha_V,\ \sin \alpha_V,\ 0\right], V \in \{A, B\} \qquad\qquad (4)$$

It is assumed that:

Domains Ω_V, $V \in \{A, B\}$ are made of two isotropic heat conductors with conductivities k_V^R, k_V^M, $V \in \{A, B\}$; the upper superscripts "R", "M" can be interpreted as being referred to certain reinforcement and matrix materials, respectively.

Domains Ω_V, $V \in \{A, B\}$, are periodically stratified in directions of Ox_V^1-axis with periods η_V, $V \in \{A, B\}$. The most important statement of the subsequent consideration is given by condition:

$$\frac{\eta_V}{\lambda_V} << 1, \quad V \in \{A, B\} \tag{5}$$

At the same time we introduce fractions of components in Ω_V setting:

$$\varphi_V^R = \frac{\eta_V^R}{\eta_V}, \quad \varphi_V^M = \frac{\eta_V^M}{\eta_V}, \quad V \in \{A, B\} \tag{6}$$

where η_V^R, η_V^M are thicknesses of homogeneous sublayers.

A fragment of the representative sublayer in a micro-periodic composite occupying domain Ω_V, $V \in \{A, B\}$ is shown in Figure 4.
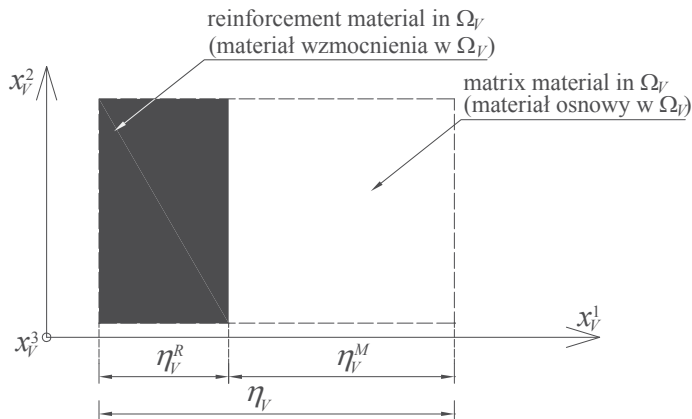

Fig. 4.   A fragment of the representative sublayer
Rys. 4.   Fragment reprezentatywnej podwarstwy

Mezzo-homogenization and mezzo-heat conductivities

Using the denotations introduced in the Previous Section we can determine the effective heat conductivities in Ω_V, $V \in \{A, B\}$.

Taking into account well-known procedure in the homogenization theory for periodically stratified composite we define the independent heat conductivities in mezzo-layers:

In the direction of Ox_V^1-axis $V \in \{A, B\}$ (i.e. in the direction normal to interfaces in Ω_V) we obtain the conductivities in the form:

$$K_V^n = \left(\frac{\varphi_V^R}{k_V^R} + \frac{\varphi_V^M}{k_V^M} \right)^{-1} \tag{7}$$

Similarly in every direction normal to Ox_V^1-axis, $V \in \{A, B\}$ (i.e. in every direction tangent to interfaces in Ω_V) these conductivities are given by:

$$K_V^t = \varphi_V^R k_V^R + \varphi_V^M k_V^M \qquad (8)$$

Formulas (7), (8) represent the effective properties of the heat conductor on the mezzo-scale.

Macro-homogenization

The main aim of this contribution is to determine the approach leading to the effective heat conductivities from mezzo to the macro effective conductivities for the biperiodically stratified composite. This can be done on the bases of formulas (7), (8) obtained for mezzo-effective conductivities in Ω_V, $V \in \{A, B\}$.

We restrict ourselves to some specific cases of this problem which will be analyzed in subsequent Section.

Obviously homogenization of biperiodically stratified composite coincides with macro-homogenization of the previously mezzo-homogenized composites.

SPECIFIC CASES

All subsequent will be carried out under extra assumption: $k_A^R = k_B^R$, $k_A^M = k_B^M$. It means that the biperiodically stratified composite under consideration is made of two isotropic materials with heat conductivities k^R and k^M.

Case 1. $n_A = n_B = [1, 0, 0]$ in $Ox^1x^2x^3$ coordinate space. If $\varphi_A = \varphi_B$ we deal with a periodically stratified space. This situation is shown in Figure 5 being nontrivial provided that $\varphi_A \neq \varphi_B$.
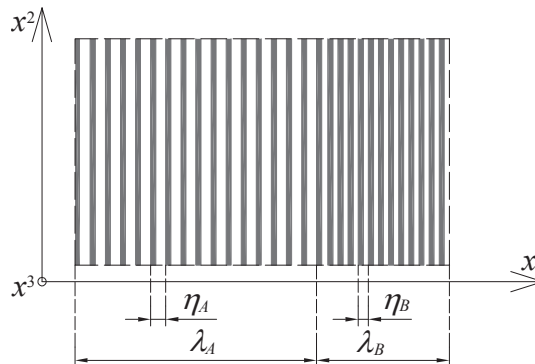

Fig. 5.   A fragment of the representative layer in Case 1
Rys. 5.   Fragment warstwy reprezentatywnej w przypadku 1

Taking into account formulas (7), (8) and applying the macro-homogenization proce-dure we obtain the following macro effective conductivities in the direction of Ox^1-axis:

$$K_0^{11} = \frac{k^R k^M}{\varphi_B \left(\varphi_B^R k^M + \varphi_B^M k^R\right) + \varphi_A \left(\varphi_A^R k^M + \varphi_A^M k^R\right)} \tag{9}$$

As well as the macro-effective conductivities in the directions of Ox^2, Ox^3- axes are equal to:

$$K_0^{22} = \varphi_A \left(\varphi_A^R k^R + \varphi_A^M k^M\right) + \varphi_B \left(\varphi_B^R k^R + \varphi_B^M k^M\right) \tag{10}$$

$$K_0^{33} = K_0^{22} = \varphi_A \left(\varphi_A^R k^R + \varphi_A^M k^M\right) + \varphi_B \left(\varphi_B^R k^R + \varphi_B^M k^M\right) \tag{11}$$

Evidently heat conductivities in every direction normal to Ox^1- axis are identical.

Due to the procedure of the macro-homogenization the result given by formulas (9), (10), (11) represent the effective heat conductivities of specified here the biperiodically stratified composite under consideration.

Case 2. $n_A = n_B = [0, 1, 0]$ in $Ox^1 x^2 x^3$ coordinate space. If $\varphi_A = \varphi_B$ we deal with a periodically stratified space. This situation is shown in Figure 6 and is nontrivial provided that $\varphi_A \neq \varphi_B$.
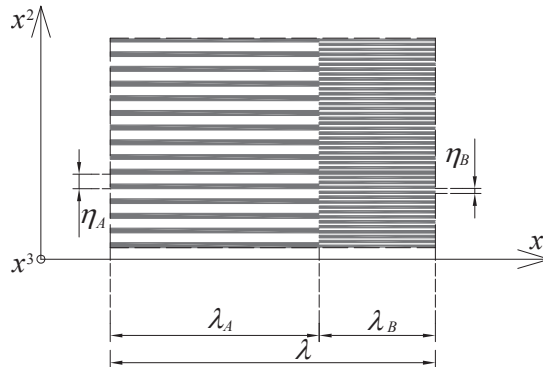

Fig. 6.   A fragment of the representative layer in Case 2
Rys. 6.   Fragment warstwy reprezentatywnej w przypadku 2

Taking into account formulas (7), (8) and applying the macro homogenization proce-dure we obtain the following macro-effective conductivities in the direction of Ox^1-axis:

$$K_0^{11} = \frac{\left(\varphi_A^R k^R + \varphi_A^M k^M\right)\left(\varphi_B^R k^R + \varphi_B^M k^M\right)}{\varphi_A \left(\varphi_B^R k^R + \varphi_B^M k^M\right) + \varphi_B \left(\varphi_A^R k^R + \varphi_A^M k^M\right)} \tag{12}$$

As well as the macro-effective conductivities in the direction to Ox^2, Ox^3-axes:

$$K_0^{22} = k^R k^M \frac{\varphi_A \left(\varphi_B^R k^M + \varphi_B^M k^R \right) + \varphi_B \left(\varphi_A^R k^M + \varphi_A^M k^R \right)}{\left(\varphi_A^R k^M + \varphi_A^M k^R \right)\left(\varphi_B^R k^M + \varphi_B^M k^R \right)} \tag{13}$$

$$K_0^{33} = \varphi_A \left(\varphi_A^R k^R + \varphi_A^M k^M \right) + \varphi_B \left(\varphi_B^R k^R + \varphi_B^M k^M \right) \tag{14}$$

Due to the procedure of the macro-homogenization the result given by formulas (12), (13), (14) represent the effective heat conductivities of special here the biperiodically stratified composite under consideration.

Case 3. $n_A = [1, 0, 0]$, $n_B = [0, 1, 0]$ in $Ox^1x^2x^3$ coordinate space. This situation is shown in Figure 7.
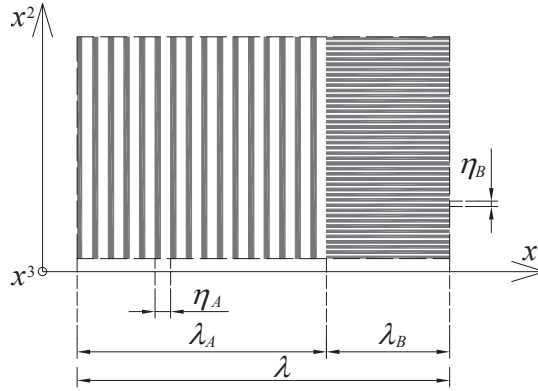

Fig. 7.   A fragment of the representative layer in Case 3
Rys. 7.   Fragment warstwy reprezentatywnej w przypadku 3

Taking into account formulas (7), (8) and applying the macro-homogenization procedure we obtain the following macro-effective conductivities in the direction of Ox^1-axis:

$$K_0^{11} = k^R k^M \frac{\left(\varphi_B^R k^R + \varphi_B^M k^M \right)}{\varphi_A \left(\varphi_B^R k^R + \varphi_B^M k^M \right)\left(\varphi_A^R k^M + \varphi_A^M k^R \right) + \varphi_B k^R k^M} \tag{15}$$

As well as in the macro-effective conductivities in the direction of Ox^2, Ox^3-axes are equal to:

$$K_0^{22} = \frac{\varphi_A \left(\varphi_A^R k^R + \varphi_A^M k^M \right)\left(\varphi_B^R k^M + \varphi_B^M k^R \right) + \varphi_B k^R k^M}{\varphi_B^R k^M + \varphi_B^M k^R} \tag{16}$$

$$K_0^{33} = \varphi_A \left(\varphi_A^R k^R + \varphi_A^M k^M \right) + \varphi_B \left(\varphi_B^R k^R + \varphi_B^M k^M \right) \qquad (17)$$

Due to the procedure of the macro-homogenization the result given by formulas (15), (16), (17) represent the effective heat conductivities of specified here biperiodically stratified composite under consideration.

CONCLUSION

The peculiarity of the procedure which makes it possible to determine the effective conductivities of biperiodically stratified heat conductors is the two-step homogenization. It is realized firstly on the mezzo-scale and secondly on the macro-scale of analysis of the composite structure under consideration.

The most important role plays here a separability of both scale given by conditions (7), (8). This separability is analogous to the well known separability of the micro- and macro-scales which constitutes the basis for applications of the asymptotic approach to the physical homogenization. Hence the simple conclusion is that the physical concept of a biperiodically stratified composite is strongly restricted by requirement $\eta_A \ll \lambda_A$, $\eta_B \ll \lambda_B$. From the point of view of engineering applications the upper bound of ration η_V / λ_V, $V \in \{A, B\}$ can be treated as equal $(\lambda_A + \lambda_B)/L$, where L is the smallest characteristic length dimension of the region occupied by the composite under consideration.

To make this contribution more simple analysis was restricted to the case in which every composite on the mezzo-scale was made of two isotropic material referred to as matrix and reinforcement. However, all considerations can be also carried out for an arbitrary periodic material structure independent in part Ω_A and Ω_B of the composite space. Consideration can be easily extended on the mezzo-scale by replacing stratification by an arbitrary periodic material structure in Ox^1-axis direction. On the other hand, investigations concerning heat conduction problems in biperiodically stratified composites can be also applied to the thermo-elastic problems.

FINAL REMARKS

At the end of this paper we have to mention that the concept of biperiodically stratified composites has been studied exclusively in the framework of homogenization. In the above mentioned papers the analysis was focused on the effect of microstructure length scale on the overall behavior of composites, c.f. [Woźniak 2002, Woźniak and Woźniak 2003, Jędrysiak and Woźniak 2006]. It means that the effect of the microstructure length on overall behavior of the composite was neglected. However the procedure applied in this paper can be generalized by applying the tolerance modelling in which the mocrostructure length scale effect is taken into account, c.f. [Jędrysiak 1999, Michalak 2001, Rychlewska 2006, 2008, Szymczyk 2006, Szymczyk and Woźniak 2006, Thermomechanics of Microheterogeneous Solids... 2008, Mathematical Modelling and Analysis... 2010].

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EFEKTYWNE WSPÓŁCZYNNIKI PRZEWODZENIA CIEPŁA DLA PEWNYCH BIPERIODYCZNYCH KOMPOZYTÓW WARSTWOWYCH

Streszczenie. Biperiodyczne warstwowe kompozyty charakteryzują się dwoma rodzajami uwarstwienia, które realizowane są niezależnie w mikro- i mezzoskali. Przykład struktury biperiodycznej pokazano na rysunku 1. Celem pracy jest zaproponowanie pewnego nowego podejścia opartego na ogólnej homogenizacji, w którego wyniku wyznaczamy efektywne współczynniki przewodzenia ciepła, oraz zilustrowanie tego podejścia dla szczególnych przypadków uwarstwienia.

Słowa kluczowe: przewodzenie ciepła, biperiodyczne materiały kompozytowe, mezzohomogenizacja, makrohomogenizacja