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TOLERANCE MODELLING OF HEAT CONDUCTION IN BIPERIODIC FOURTH-COMPONENT COMPOSITE*

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ABSTRACT

The object of analysis is a heat conduction problem within the framework of tolerance modelling in fourth--component biperiodic composite. Two materials are isotropic and two are orthotropic, and additionally symmetry axes of orthotropic ones are rotated with respect to each other by an angle equal 90°. The results of some special boundary conditions for stationary problem of heat conduction were obtained from the local homogenization model (LHM). The model equations were derived by simplifying the equations of the standard tolerance model (STM), which were obtained based on two model assumptions: micro-macro decomposition of the temperature field and residual function averaging, after introducing the concept of "weakly slowly varying function" (WSV) and "slowly varying function" (SV) into the modelling process. The presented examples show the influence of the given boundary conditions on the macro-temperature distribution and on the distribution of the approximate temperature field $\theta(\cdot)$. The effect of thermal conductivity of the component materials and the number of periodicity cells on temperature distribution was also shown.

Key words: biperiodic composite, tolerance modelling, heat conduction

INTRODUCTION

The problem of heat conduction in periodic layered composites has been well known and extensively described in the literature. The basic difficulty in modelling the problem of heat conductivity in composites with unidirectional periodicity is the use of such a technique that will allow for the replacement of highly oscillating coefficients in the partial differential equations of heat conduction with coefficients that will have constant values.

One of the methods of modelling this type of structure is asymptotic homogenization introduced by Jikov, Kozlov and Oleinik (1994). There are also a number of papers in the literature on obtaining averaged characteristics of material properties for periodic composites in a different way. One can mention the method proposed by Woźniak (1987), called the method of microlocal parameters. Another methods are tolerance averaging technique and the asymptotic variant of tolerance modelling called local homogenization (Woźniak & Wierzbicki, 2000; Woźniak, Michalak & Jędrysiak, 2008; Awrajcewicz et al., 2010). The above-mentioned models make it possible to describe and determine approximate solutions of heat conduction and elastostatics problem for two-component layered composites with a periodic structure or with a structure with a functional gradation of effective properties.

In the case of bi- or three-directional periodic composites, introducing an averaged model is possible if certain additional conditions for the construction of this composite are met (Jikov et al., 1994). This means that only for a certain class of bi- or three-directional periodic composites it is possible to determine the model equations within

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the framework of asymptotic homogenization, tolerance modelling (tolerance averaging technique) or local homogenization. For bidirectionally periodic composites, in which the periodicity cell is built of four different materials, homogenization is possible, for example, when two materials are isotropic and two are orthotropic, and additionally orthotropic materials have correspondingly arranged axes of symmetry. This was proven by Czarnecka (2014). The material properties of the components and their alignment must ensure that the composite behaves in a streaked manner. The analysis of heat conductivity problems in composites constructed in this way can be found, for example, in the works by Szlachetka, Wągrowska and Woźniak (2013), Czarnecka (2014), Wągrowska and Woźniak (2014), Wągrowska and Szlachetka (2015), Kubacka and Ostrowski (2021).

MATERIAL AND METHODS

The subject of the considerations is the issue of heat conduction in rigid, bidirectional periodic composites.

The considered composite occupies the area in three-dimensional Cartesian space $\Omega \equiv \Xi \times (0, L_3)$, where $\Xi = (0, L_1) \times (0, L_2)$. Let denote by $\boldsymbol{x} \equiv (x_1, x_2, x_3)$ the points belonging to the area Ω, where $x_1 \in (0, L_1),\ x_2 \in (0, L_2),\ x_3 \in (0, L_3)$. It means that the composite under consideration is a biperiodic composite in the plane Ox_1x_2, with periodical periods in the direction of the axis Ox_1 and Ox_2 equal to η_1 and η_2, respectively. The composite scheme is shown in Figure 1.

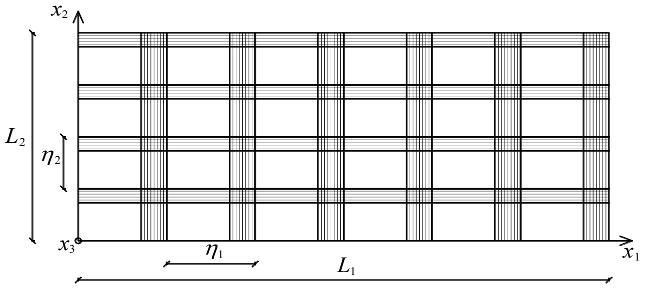

Fig. 1. Scheme of the composite (Wągrowska & Szlachetka, 2015)

It should be emphasized that η_α where $\alpha = 1, 2$, is a positive parameter, i.e. $\eta_\alpha > 0$, which is much smaller than the dimension of the characteristic area Ω, i.e. $\eta_\alpha << L_\alpha$.

Let denote the points $\left(x_{1i}, x_{2j}\right) \equiv \left(\frac{1}{2}\eta_1 + (i-1)\eta_1, \frac{1}{2}\eta_2 + (j-1)\eta_2\right)$, where $i = 1, \ldots, n; j = 1, \ldots, m$. Thus the periodic cell can be defined as $\Delta\left(x_{1i}, x_{2j}\right) \equiv \left(x_{1i} - \frac{1}{2}\eta_1, x_{1i} + \frac{1}{2}\eta_1\right) \times \left(x_{2j} - \frac{1}{2}\eta_2, x_{2j} + \frac{1}{2}\eta_2\right)$.

A scheme of a repeating cell is shown in Figure 2.

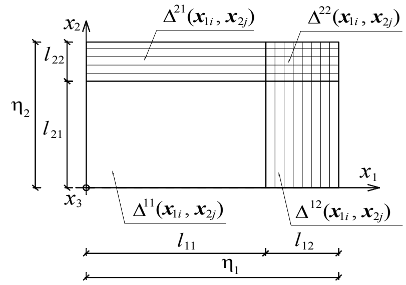

Fig. 2. Diagram of the periodicity cell (Wągrowska & Szlachetka, 2015)

It was assumed here as in Czarnecka (2014) and Kubacka and Ostrowski (2021) that the periodicity cell is built of four materials: two isotropic and two orthotropic, of which the orthotropic ones are characterized by the fact that their symmetry axes are rotated by an angle equal 90°.

The considered composite has a biperiodic structure (periodic in two different directions) this implies that the structure can be defined by a certain functions $\varphi_{\alpha 1} = \varphi_{\alpha 1}(\boldsymbol{x})$, $\alpha = 1, 2$ (Woźniak et al., 2008), which describes the fractions of the materials. This functions are a differentiable functions, defined in area $\boldsymbol{\Xi}$, such that $0 < \varphi_{\alpha 1} < 1$ which are slowly varying in the area Ω occupied by the composite. Functions $\varphi_{\alpha 1}$ can be understood as a functions of the saturation of the composite with a given material.

It follows that the dimensions of the areas occupied by a given material in the periodic cell can be written as: $l_{\alpha 1} \equiv \varphi_{\alpha 1}\eta_\alpha$, $l_{\alpha 2} \equiv \varphi_{\alpha 2}\eta_\alpha$ where $\varphi_{\alpha 2} \equiv 1 - \varphi_{\alpha 1}$ and $l_{\alpha 1} + l_{\alpha 2} = \eta_\alpha$ for each $\boldsymbol{x}$.

Matrices of heat conduction coefficients $\boldsymbol{K}^{hl}$ and specific heat c^{hl} in a given cell $\Delta^{hl}\left(x_{1i}, x_{2j}\right) i, j = 1,2,$ h, $l = 1,2$ take the form (Czarnecka, 2014):

$$\boldsymbol{K}^{11} = \begin{bmatrix} k^{11} & 0 \\ 0 & k^{11} \end{bmatrix}, \ \boldsymbol{K}^{12} = \begin{bmatrix} k^{11} & 0 \\ 0 & k^{22} \end{bmatrix}, \ \boldsymbol{K}^{21} = \begin{bmatrix} k^{22} & 0 \\ 0 & k^{11} \end{bmatrix}, \ \boldsymbol{K}^{22} = \begin{bmatrix} k^{22} & 0 \\ 0 & k^{22} \end{bmatrix},$$

$$c^{11} = c^{21} = c^{12} = c^{22}.$$

The Fourier equation for the temperature field $\theta = \theta(\boldsymbol{x}, t)$, assuming that there are no heat sources acting in the area occupied by the considered composite, can be written as:

$$\boldsymbol{c}\partial_t \theta - \partial_\alpha \left(k^{\alpha\beta}\partial_\beta \theta\right) = 0, \quad \alpha, \beta = 1,2, \tag{1}$$

where:

$\theta(\cdot,\cdot) - \theta$ – temperature in the area Ω for each time $t \in [0, t_*)$,

$\partial_\beta \equiv \dfrac{\partial}{\partial x^\beta}$, $\partial_t \equiv \dfrac{\partial}{\partial t}$, $c(\cdot)$ – specific heat,

$k^{\alpha\beta}(\cdot)$ – thermal conductivity coefficients such that $k^{\alpha\beta}(\cdot) = 0$ for $\alpha \neq \beta$.

Equation (1) is a partial differential equation with discontinuous and highly oscillating coefficients.

The process of tolerance modelling of heat conductivity in multicomponent biperiodic composites is based on the concept of slow-varying functions, tolerance averaging approximation and oscillating shape function. Individual definitions of these terms will be provided in the next semi-chapter.

Slowly varying functions

The concept of a slowly varying function (Woźniak et al., 2008; Awrajcewicz et al., 2010) is closely related to the concept of tolerance.

The tolerance relation is defined as a binary relation – reflexive and symmetric. An example of such a relation between points in space $R \times R$ is the relation of indistinctness with the parameter of tolerance (δ). This means that each pair of numbers such $(\mu, \nu) \in R^2$ – is within tolerance δ (which can be symbolically written: $\mu \overset{\delta}{\sim} \nu$) if and only if the norm from the difference μ and ν is less than or equal to δ, i.e. $|\mu - \nu| \leq \delta$. The tolerance relationship defined in this way can be found in monographs (Woźniak et al., 2008; Awrajcewicz et al., 2010).

In Woźniak et al. (2008), Awrajcewicz et al. (2010) and Woźniak, Wągrowska and Szlachetka (2015) there were introduced two classes of slowly varying and differentiable functions. These functions are called "weakly slowly varying functions" and "slowly varying functions", and denoted as WSV and SV, respectively.

Let Π stand for an arbitrary convex set in the space R^m and $f \in C^1(\Pi)$ be an arbitrary real-valued function. Let also introduce the tolerance parameter $d \equiv (\lambda, \delta_0, \delta_1)$ as a triplet of real positive numbers and use the notation $\partial_j \equiv \frac{\partial}{\partial x_j}$, $j = 1, ..., m$.

Function $f \in \mathrm{WSV}_d^1(\Pi) \subset C^1(\Pi)$ if the condition $\|x-y\| \le \lambda$ yields the conditions $\left(|f(x)-f(y)| \le \delta_0 \text{ and } |\partial_j f(x) - \partial_j f(y)| \le \delta_1\right)$ for $j = 1, \ldots, m$ for all pairs $(x, y) \in \Pi^2$. If $f \in \mathrm{WSV}_d^1(\Pi)$ additionally meets the condition $\lambda|\partial_j f(x)| \le \delta_0$ for $j = 1, \ldots, m$ for every $x \in \Pi$, then $f \in \mathrm{SV}_d^1(\Pi)$. Obviously, this means that $\mathrm{SV}_d^1(\Pi) \subset \mathrm{WSV}_d^1(\Pi)$.

Tolerance averaging approximation

Let determine that $\Delta(\boldsymbol{x})$ for $\boldsymbol{x} \in R^2$ be a periodic cell with center on point $\boldsymbol{x} \in R^2$ of the biperiodic composite.

Define also $\Omega_0 \subset \Omega$ as:

$$\Omega_0 \equiv \{x \in \Omega: \quad \Delta(\boldsymbol{x}) \in \Omega\}.$$

For any function $g(\cdot)$ integrable with square on the set $\Omega(g \in L^2(\Omega))$, the averaging of this function at a point $\boldsymbol{x} \in \Omega_0$ is equal:

$$\langle g \rangle(x) \equiv \frac{1}{\eta_1 \eta_2} \int_{\Delta(x)} g(\boldsymbol{y}) \, dy_1 dy_2 . \tag{2}$$

When $g \in L^2(\Omega)$ and $f \in \mathrm{WSV}_d^1(\Pi)$, then the tolerance averaging approximation is understood as the approximation of the functions: $\langle gf \rangle_T(x)$ and $\langle g\partial f \rangle_T$ by the functions $\langle g \rangle(x) f(x)$ and $\langle g \rangle(x) \partial f(x)$, respectively.

Fluctuation shape function

Let introduce $(x_1 i^-, x_1 i^+)$, (x_{2j}^-, x_{2j}^+) as an arbitrary pairs of planes in the space R^2 which there are parallel interfaces between individual cells of periodicity:

$$x_{1i}^- = x_{1i} - \frac{1}{2}\eta_1, \, x_{1i}^+ = x_{1i} + \frac{1}{2}\eta_1, \, x_{2j}^- = x_{2j} - \frac{1}{2}\eta_2, \, x_{2j}^+ = x_{2j} + \frac{1}{2}\eta_2, \, i = 1, \ldots, n, \;\; j = 1, \ldots, m.$$

The plane separating the materials in a cell is defined as $\tilde{x}_\alpha = x_\alpha^- + l_{\alpha 1} = x_\alpha^+ - l_{\alpha 2}$, where $x_\alpha^- = x_{1i}^-$ or x_{2j}^- and $x_\alpha^+ = x_{1i}^+$ or x_{2j}^+.

The real-valued function $\gamma^{\alpha} \in C^{0}\left(\left[0, L_{\alpha}\right]\right)$ where $\alpha = 1, 2$ are said to be fluctuation shape functions. They are linear continuous functions in the intervals depending only on one argument and taking values on individual interfaces as follow: $\gamma^{\alpha}\left(x_{\alpha}^{-}\right) = -\frac{\eta_{\alpha}}{2}$, $\gamma^{\alpha}\left(x_{\alpha}^{+}\right) = -\frac{\eta_{\alpha}}{2}$, $\gamma^{\alpha}\left(\bar{x}_{\alpha}\right) = \frac{\eta_{\alpha}}{2}$. The scheme of the fluctuation shape function is shown in Figure 3.

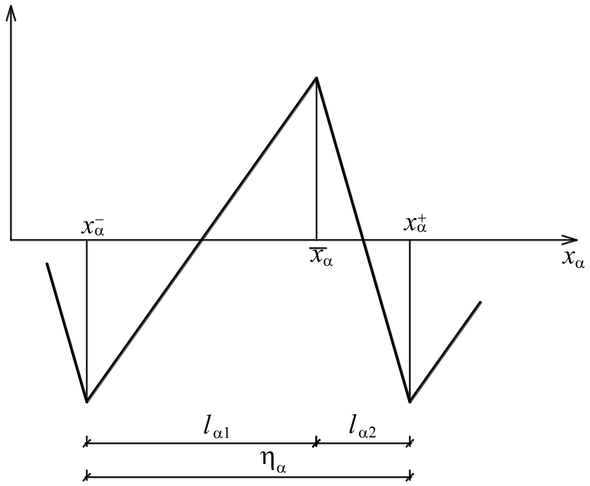

Fig. 3. Scheme of the fluctuation shape function (Wągrowska & Szlachetka, 2015)

Modelling procedure

The tolerance modelling procedure is based on two assumptions. In the first assumption called micro-macro decomposition of the temperature field, the temperature function $\theta(\cdot)$ is approximated by the function $\tilde{\theta}(\cdot)$ in the form (Czarnecka, 2014; Wągrowska & Woźniak, 2014):

$$\theta(\boldsymbol{x}, t) \approx \tilde{\theta}(\boldsymbol{x}, t) = \vartheta(\boldsymbol{x}, t) + \gamma^{1}(x_1)\psi_1(\boldsymbol{x}, t) + \gamma^{2}(x_2)\psi_2(\boldsymbol{x}, t), \tag{3}$$

where for every $x_3 \in (0, L_3)$ fields:

- $\vartheta(\cdot, x_3, t)$ and $\psi_{\alpha}(\cdot, x_3, t)$, $\alpha = 1, 2$ are the unknown functions called macro-temperature and amplitudes of temperature fluctuations, respectively.
- Functions $\gamma^{\alpha}(\cdot)$, $\alpha = 1, 2$, are fluctuation shape functions introduced in the previous semi-chapter.

In order to formulate the second modelling assumption, let us introduce residual function:

$$r_x(\cdot) \equiv \partial_{\alpha}\left(k_x^{\alpha\beta}(\cdot)\partial_{\beta}\tilde{\theta}(\cdot)\right) - c_x \partial_t \tilde{\theta}(\cdot). \tag{4}$$

Residual function is defined almost everywhere in $\Omega \times [0, t_*)$ and $\tilde{\theta}(\cdot)$ is defined by Equation (3). The second assumption in modelling procedure can be formulated as follows: $\langle r \rangle_T = 0$, $\langle rh \rangle_T = 0$, $\vee (x_1, x_2) \in \Xi$. Obviously, the above-introduced tolerance averages are also defined for all (x_3, t). In tolerance modelling, in general tolerance model (GTM), about the functions $\vartheta(\cdot, x_3, t)$ and

$\psi_\alpha(\cdot, x_3, t)$, $\alpha = 1, 2$ it is assumed that they are weakly slowly varying functions, i.e. $\vartheta(\cdot, x_3, t)$, $\psi_\alpha(\cdot, x_3, t) \in \mathrm{WSV}_d^1((0, L_1)\times(0, L_2))$ for each x_3 and $t \in [0, t_*)$ (Nagórko & Woźniak, 2011; Czarnecka, 2014; Woźniak, Wągrowska & Szlachetka, 2015).

In this paper it was assumed that the unknown functions are slow-varying functions, i.e. $\vartheta(\cdot, x_3, t)$, $\psi_\alpha(\cdot, x_3, t) \in \mathrm{SV}_d^1((0, L_1)\times(0, L_2))\forall(x_3, t)$. This assumption leads to the standard tolerance model (STM) equations. The equations of the STM for the stationary problem and in the absence of heat sources acting in the area occupied by the considered composite take the form:

$$\langle k^{\alpha\beta}\rangle \partial_\alpha \partial_\beta \vartheta + \partial_\alpha\left(\langle k^{\alpha\beta}\chi_\beta^\delta\rangle \psi_\delta\right) - \langle c\rangle \partial_t \vartheta = 0, \tag{5a}$$

$$\left\langle k^{11}\left(\gamma^1\right)^2\right\rangle \partial_1^2\psi_1 - \left\langle k^{11}\left(\partial_1\gamma^1\right)^2\right\rangle \psi_1 - \langle k^{11}\partial_1\gamma^1\rangle \partial_1\vartheta - \left\langle c\left(\gamma^1\right)^2\right\rangle \partial_t\psi_1 = 0, \tag{5b}$$

$$\left\langle k^{22}\left(\gamma^2\right)^2\right\rangle \partial_2^2\psi_2 - \left\langle k^{22}\left(\partial_2\gamma^2\right)^2\right\rangle \psi_2 - \langle k^{22}\partial_2\gamma^2\rangle \partial_2\vartheta - \left\langle c\left(\gamma^2\right)^2\right\rangle \partial_t\psi_2 = 0. \tag{5c}$$

where: $\chi_\beta^\delta = \partial_\beta \gamma^\delta$.

Based on Woźniak et al. (2015) let assume that $\vartheta(\cdot, x_3, t) \in \mathrm{SV}_d^1((0, L_1)\times(0, L_2))\forall(x_3, t)$ and $\psi_\alpha(\cdot) \in \mathrm{SV}_d^1(\Omega\times(0, t_*))$ with the extra assumption that terms $\left\langle c\left(\gamma^1\right)^2\right\rangle \partial_t\psi_1$, $\left\langle c\left(\gamma^2\right)^2\right\rangle \partial_t\psi_2$ in (4) can be neglected.

In this case Equation (5a) remains unchanged. Terms $\left\langle k^{11}\left(\gamma^1\right)^2\right\rangle \partial_1^2\psi_1$, $\left\langle k^{22}\left(\gamma^2\right)^2\right\rangle \partial_2^2\psi_2$ vanish because $\psi_\alpha(\cdot) \in \mathrm{SV}_d^1(\Omega\times(0, t_*))$. And neglecting terms $\left\langle c\left(\gamma^1\right)^2\right\rangle \partial_t\psi_1$, $\left\langle c\left(\gamma^2\right)^2\right\rangle \partial_t\psi_2$ we obtain the equations of the local homogenization model (LHM) (Nagórko & Woźniak, 2011; Woźniak et al., 2015):

$$\begin{gathered}\langle k^{\alpha\beta}\rangle \partial_\alpha \partial_\beta \vartheta + \partial_\alpha\left(\langle k^{\alpha\beta}\chi_\beta^\delta\rangle \psi_\delta\right) - \langle c\rangle \partial_t \vartheta = 0,\\ \left\langle k^{11}\left(\partial_1\gamma^1\right)^2\right\rangle \psi_1 + \langle k^{11}\partial_1\gamma^1\rangle \partial_1\vartheta_1 = 0,\\ \left\langle k^{22}\left(\partial_2\gamma^2\right)^2\right\rangle \psi_2 + \left\langle k^{22}\left(\partial_2\gamma^2\right)^2\right\rangle \partial_2\vartheta = 0.\end{gathered} \tag{6}$$

This paper deals with stationary heat conductivity problem. It involves, that the term $\langle c\rangle \partial_t \vartheta_1$ vanishes and Equations (6) take the form:

$$\langle k^{\alpha\beta}\rangle \partial_\alpha \partial_\beta \vartheta + \partial_\alpha\left(\langle k^{\alpha\beta}\chi_\beta^\delta\rangle \psi_\delta\right) = 0, \tag{7a}$$

$$\left\langle k^{11}\left(\partial_1\gamma^1\right)^2\right\rangle \psi_1 + \langle k^{11}\partial_1\gamma^1\rangle \partial_1\vartheta_1 = 0, \tag{7b}$$

$$\left\langle k^{22}\left(\partial_2\gamma^2\right)^2\right\rangle \psi_2 + \langle k^{22}\partial_2\gamma^2\rangle \partial_2\vartheta = 0. \tag{7c}$$

It follows from (7b) and (7c) that:

$$\psi_1 = -\frac{\langle k^{11}\partial_1\gamma^1\rangle}{\left\langle k^{11}\left(\partial_1\gamma^1\right)^2\right\rangle}\partial_1\vartheta,\ \ \psi_2 = -\frac{\langle k^{22}\partial_2\gamma^2\rangle}{\left\langle k^{22}\left(\partial_2\gamma^2\right)^2\right\rangle}\partial_2\vartheta. \tag{8}$$

In view of Relationships (8), Equation (7a) yields to:

$$k_{eff}^{11}\partial_1\partial_1\vartheta + k_{eff}^{22}\partial_2\partial_2\vartheta = 0,\ \ x\in\Omega\times(0, L_3), \tag{9}$$

where:

$$k_{eff}^{11} \equiv \left\langle \frac{1}{k^{11}(x_1)} \right\rangle^{-1},\quad k_{eff}^{22} \equiv \left\langle \frac{1}{k^{22}(x_2)} \right\rangle^{-1}. \tag{10}$$

Equation (9) and Relationships (8) along with the boundary conditions for the searched function $\vartheta(\cdot)$ are analogous to the equations obtained as a result of applying the asymptotic approach by Czarnecka (2014). The same solution was achieved by Kubacka and Ostrowski (2021) using tolerance modelling based on Eulerian-Lagrangian equations. It should be emphasized that Equation (9) obtained as a result of the tolerance modelling process, assuming that $\vartheta(\cdot, x_3, t)\in \mathrm{SV}_d^1\left((0, L_1)\times(0, L_2)\right)\forall(x_3, t)$ and $\psi_\alpha(\cdot)\in \mathrm{SV}_d^1\left(\Omega\times(0, t_*)\right)$ for stationary problems has constant coefficients $\left(k_{eff}^{11}, k_{eff}^{22}\right)$ as opposed to Equation (1).

After solving the correctly posed boundary problem for $\vartheta(\cdot)$ and determining the $\psi_\alpha(\cdot)$, $\alpha = 1$, 2 form (8) the temperature field $\tilde{\theta}(\cdot)$ in a biperiodic composite should be determined from Equation (3).

Boundary conditions

Let assume the boundary conditions on macro-temperature $\vartheta(\cdot)$ in form: $\vartheta(x_1, 0) = a_1(x_1)$, $\vartheta(x_1, L_2) = a_2(x_1)$, $\vartheta(0, x_2) = b_1(x_2)$, $\vartheta(L_1, x_2) = b_2(x_2)$, solved using the method of separation of variables (Bagdasaryan & Chalecki, 2016; Zill, 2016).

Then the solution for $\vartheta(\cdot)$ takes the form:

$$\begin{aligned}\vartheta(x_1, x_2) = {} & \frac{2}{L_1}\sum_{n=1}^{\infty}\left(A_n \sinh\left[\kappa\frac{n\pi(L_2 - x_2)}{L_1}\right] + B_n \sinh\left[\kappa\frac{n\pi x_2}{L_1}\right]\right)\times \\ & \times\sin\left[\frac{n\pi x_1}{L_1}\right]\cdot\left(\sinh\left[\kappa\frac{n\pi L_2}{L_1}\right]\right)^{-1} + \frac{2}{L_2}\sum_{n=1}^{\infty}\left(C_n\sinh\left[\kappa\frac{n\pi(L_1 - x_1)}{L_2}\right] + \right. \\ & \left. + D_n\sinh\left[\kappa\frac{n\pi x_1}{L_2}\right]\right)\cdot\sin\left[\frac{n\pi x_2}{L_2}\right]\cdot\left(\sinh\left[\kappa\frac{n\pi L_1}{L_2}\right]\right)^{-1},\end{aligned} \tag{11}$$

where:

$$\begin{aligned}&\kappa = \frac{k_{eff}^{11}}{k_{eff}^{22}},\ A_n = \int_0^{L_1} a_1(x_1)\sin\left[\frac{n\pi x_1}{L_1}\right]dx_1,\ B_n = \int_0^{L_1} a_2(x_1)\sin\left[\frac{n\pi x_1}{L_1}\right]dx_1, \\ &C_n = \int_0^{L_2} b_1(x_2)\sin\left[\frac{n\pi x_2}{L_2}\right]dx_2,\ D_n = \int_0^{L_2} b_2(x_2)\sin\left[\frac{n\pi x_2}{L_2}\right]dx_2.\end{aligned} \tag{12}$$

Solution for macro-temperature can be solved also using for example finite difference method (Kubacka & Ostrowski, 2021).

RESULTS

Let consider the four-component biperiodic structure of the composite. The composite occupied area $\Omega = (0, L_1) \times (0, L_2) \cdot R$. Let $L_1 = L_2 = 1$ [m], $\eta_1 = \frac{L_1}{6}$, $\eta_2 = \frac{L_2}{3}$. Let assume that the components are homogeneous, where two of them are isotropic and two are orthotropic with a components of the matrix of thermal conductivity coefficients equal: $k_{11} = 0.25$ W·m^{-1}·K^{-1} and $k_{22} = 0.5$ W·m^{-1}·K^{-1}. In examples assume that the saturation functions are equal $\varphi_{11} = 0.6$, $\varphi_{21} = 0.2$ then $\varphi_{12} = 1 - \varphi_{11} = 0.4$ and $\varphi_{22} = 1 - \varphi_{21} = 0.8$.

Let determine the approximate temperature distribution in the considered conductor, taking 20 terms in the series (11) and under the macro-temperature boundary conditions in the form:

Example 1:

$$a_1(x_1) = \sin(\pi x_1),\ a_2(x_1) = -\sin(\pi x_1),$$
$$b_1(x_2) = -\sin(\pi x_2),\ b_2(x_2) = \sin(\pi x_2). \qquad (13)$$

Example 2:

$$a_1(x_1) = x_1^2 - x_1,\ a_2(x_1) = 0,$$
$$b_1(x_2) = 0, \qquad b_2(x_2) = 0. \qquad (14)$$

The distribution of the approximate temperature field is $\tilde{\theta}(\cdot)$ in analysed examples are shown in Figure 4, while Figure 5 shows the contour plot of this distributions.

To observe the effect of the number of cells periodicity, in Example 2 – changed the value of $\eta_1 = \frac{L_1}{6}$ to $\eta_1 = \frac{L_1}{2}$. The result of macro-temperature distribution in 3D and in contour plot in this case is show on Figure 6.

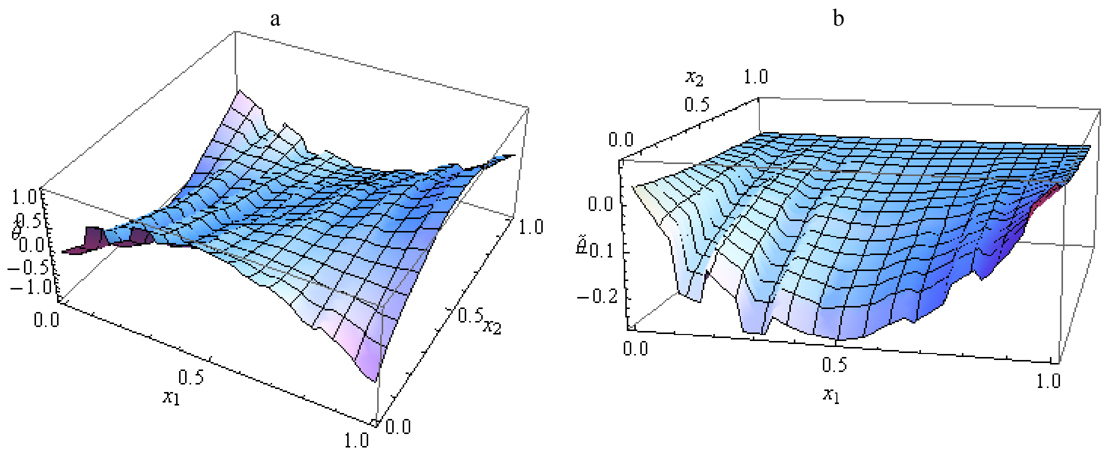

Fig. 4. Temperature $\tilde{\theta}(\cdot)$ distribution: a – Example 1; b – Example 2

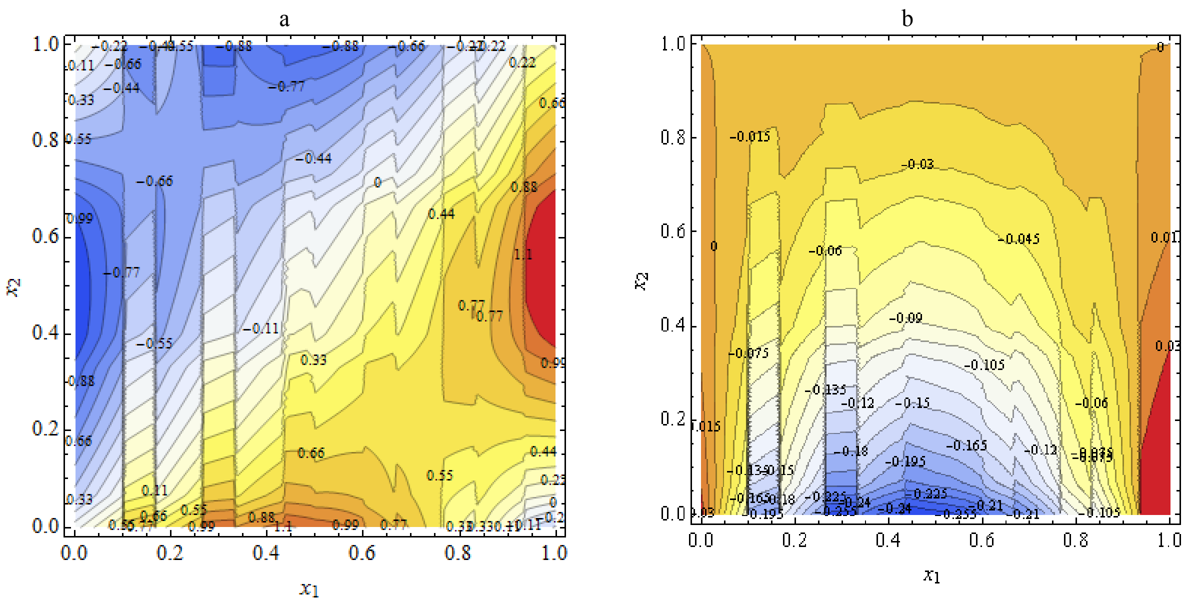

Fig. 5. Temperature $\tilde{\theta}(\cdot)$ contour plot: a – Example 1; b – Example 2

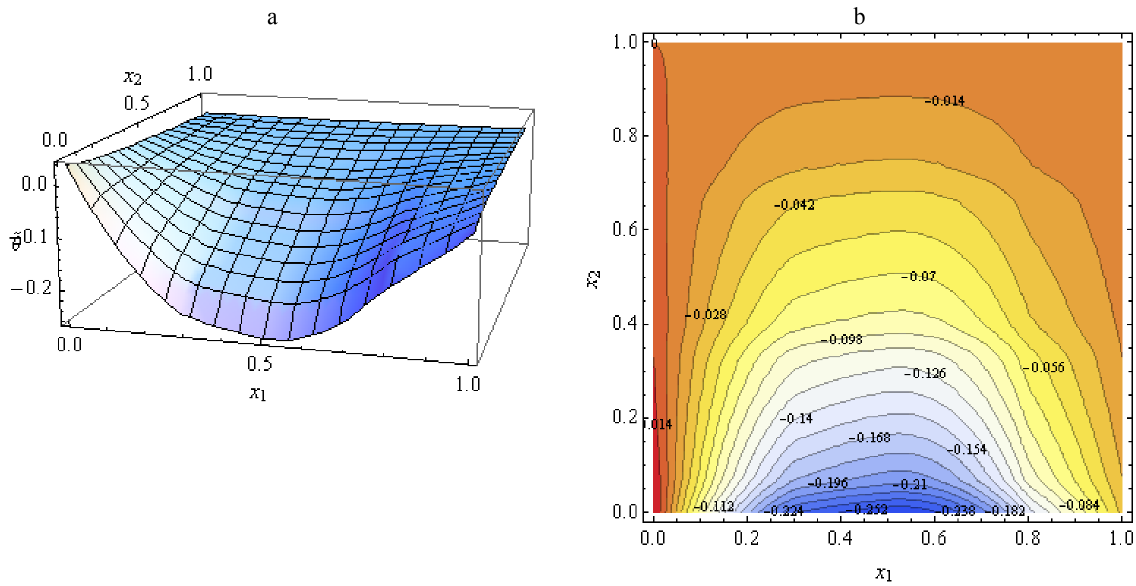

Fig. 6. Temperature $\tilde{\theta}(\cdot)$ for Example 2 with assumption that $\eta_1 = \dfrac{L_1}{2}$: a – distribution in 3D; b – contour plot

Apparently, it might appear that graph of macro-temperature for smaller number of cells is smoother than for larger number, but note that the kinks in the graph are where the different materials connect – that is, the less interfaces, the less kink in the graph. This smoothness should not be confused with the convergence of the macro--temperature graph to the micro-temperature graph. It is clear that the more the number of cells tends to infinity the macro-temperature graph tends to the micro-temperature graph (Fig. 7).

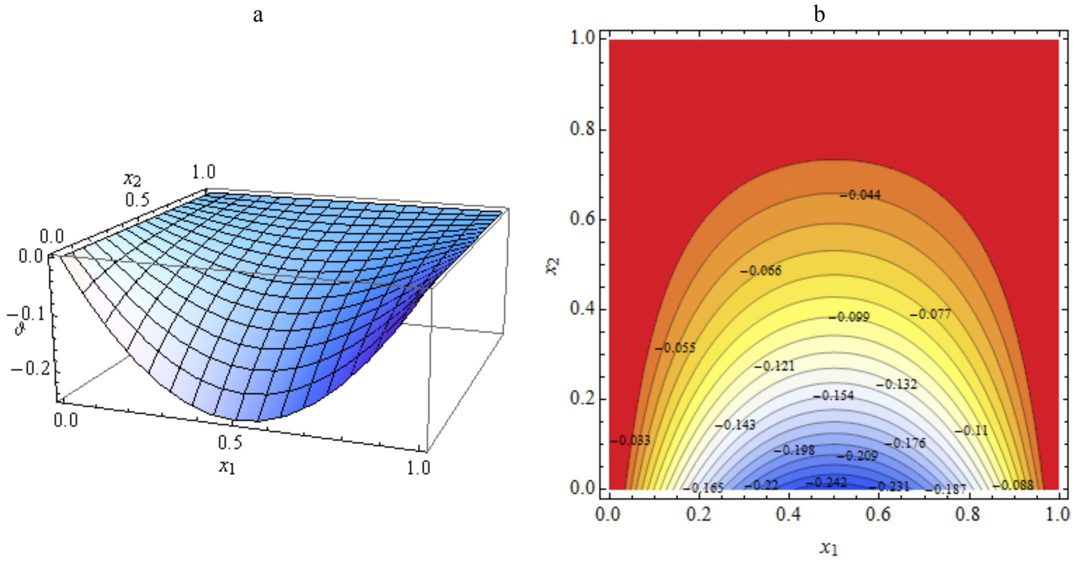

Fig. 7. Macro-temperature $\vartheta(\cdot)$ for Example 2: a – distribution in 3D; b – contour plot

CONCLUSIONS

- From the standard tolerance model (STM), the local homogenization model (LHM) equations can be derived be neglected in STM terms $\left\langle c(y^1)^2 \right\rangle \partial_t \psi_1$, $\left\langle c(y^2)^2 \right\rangle \partial_t \psi_2$.
- The form of LHM equations are analogous to the equations that can be derived using the assumptions of asymptotic homogenization.
- The macro-temperature Equation (9) with known boundary conditions can be solved by the variable separation method.
- The presented examples show the influence of the given boundary conditions on the macro-temperature distribution and on the distribution of the approximate temperature field $\tilde{\theta}(\cdot)$.
- On the distribution of the approximate temperature field $\tilde{\theta}(\cdot)$ influence accept the values of heat conductivity coefficients, the number of periodicity cells.

Authors' contributions

Conceptualisation: O.S.; methodology: O.S.; formal analysis: O.S. and J.W.D.; investigation: O.S. and J.W.D.; data curation: O.S.; writing – original draft preparation: O.S.; writing – review and editing: O.S. and J.W.D.; visualisation: O.S. and J.W.D.

All authors have read and agreed to the published version of the manuscript.

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MODELOWANIE TOLERANCYJNE PRZEWODZENIA CIEPŁA W CZTEROSKŁADNIKOWYM KOMPOZYCIE BIPERIODYCZNYM

STRESZCZENIE

Przedmiotem analizy jest problem przewodnictwa ciepła w ramach modelowania tolerancyjnego. W artykule przedstawiono rozwiązanie pewnego szczególnego problemu brzegowego dla stacjonarnego zagadnienia przewodnictwa ciepła w kompozytach w czteroskładnikowych sztywnych kompozytach biperiodycznych. Dwa materiały są izotropowe i dwa są ortotropowe, a dodatkowo osie symetrii materiałów ortotropowych są obrócone względem siebie o kąt równy 90°. Rozwiązania pewnych szczególnych warunków brzegowych dla stacjonarnego zagadnienia przewodzenia ciepła wyznaczono z modelu lokalnej homogenizacji (LHM). Równania modelu uzyskano, upraszczając równania standardowego modelu tolerancyjnego (STM), które otrzymano na podstawie dwóch założeń modelowych: mikro-makro dekompozycji pola temperatury i uśredniania funkcji rezydualnej, po wprowadzeniu do procesu modelowania koncepcji „funkcji słabo wolnozmiennej" (WSV) i „funkcji wolnozmiennej" (SV). Przedstawione przykłady pokazują wpływ zadanych warunków brzegowych na rozkłady mikrotemperatury oraz przybliżonego pola temperatury $\theta(\cdot)$. Widoczny jest również wpływ przewodności cieplnej materiałów składowych oraz liczby komórek periodyczności na rozkład temperatury.

Słowa kluczowe: kompozyt biperiodyczny, modelowanie tolerancyjne, przewodnictwo ciepła