

## ELASTIC BEAM RESTING ON VLASOV ELASTIC FOUNDATION AND SUBJECTED TO AN EXTERNAL CONCENTRATED LOAD

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### ABSTRACT

The paper presents the problem of bending of a beam uniformly loaded along its entire length, resting freely on an elastic Vlasov foundation with an additional external load  $G$  applied to the ground at a distance  $a$  from the beam end. The given example is a special case of a beam resting freely on an elastic foundation, which is present in many cases of building practice. The paper considers an approximate solution for the influence of an additional load  $G$  on the deflection and cross-sectional forces of a beam resting freely on Vlasov elastic subsoil. Graphs of a deflection of the beam and of a soil surface layer beyond the limits of the beam, as well as graphs of bending moments and transverse forces in the beam have been presented. The influence of an additional load applied in a distance  $a$  on the deflection and cross-sectional forces in the beam has been investigated.

**Key words:** elastic beam, Vlasov foundation, cross-sectional forces, beam deflection, displacement disappearance function

### INTRODUCTION

The problem of calculating structures resting on a subsoil has been known in mechanics for a long time. In this respect, there are a large number of computational models and a huge number of works using these models. Such building structures as foundation beams are very widely used in structural engineering, for example as basic elements of roads, bridges and building foundations (Yue, 2021). These structures occupy a significant place in the overall volume of construction and the cost of their construction is high. Therefore, the improvement in calculation results is reflected in the economics of building materials and in turn in the value of a construction (Leontiev et al., 1982). A huge amount of research has been done in this area, but the existing calculation methods are not perfect. The hypotheses describing displacements in

the subsoil are also not fully satisfactory. In many cases, obtained results differ significantly from the real ones (Vlasov & Leontiev, 1960). If the behavior of a beam under load can be described well enough by the Timoshenko or Bernoulli-Euler beam theory, the mechanical properties of the subsoil and its interaction with the beam are very difficult to modeling. (Avramidis & Morfidis, 2005). Two methods are commonly used to design and analyze the problems of beams resting on elastic or viscoelastic ground – analytical and numerical. Numerical methods are time-consuming and their accuracy is related to the integrated algorithm (Miao, Shi, Wang & Zhong, 2017). Analytical solutions of various problems can serve as benchmarks for testing numerical methods, although the current state of mathematics allows for finding exact solutions for a limited number of cases. However, in the case of boundary value problem concerning the elastic

half-space in analytical-numerical solutions, difficulties arise both at the stage of symbolic calculations and numerical integration (Jemioło & Szwed, 2017).

Typically, structures are made of homogeneous and isotropic materials. Their contact with the ground, for any combination of external loads, is assumed to be continuous – it is assumed that vertical displacements of the structure and the ground over the entire contact area are the same.

Unlike the Winkler model, the classical Vlasov model uses a continuum theory on the basis of the variational principle, as a result, it has a reliable theoretical basis, and the characteristic constants in the model can be expressed in terms of material properties (Liu & Ma, 2013). With the characteristic constants  $k$  and  $t$ , the compressive and shear work of the elastic foundation is taken into account, respectively.

## TWO-PARAMETER MODEL OF THE ELASTIC VLASOV SOIL

Two-dimensional models of elastic foundation are divided into two groups:

- 1) models resulting from the equations of the theory of elasticity after introducing certain simplifications – they are called structural models,
- 2) models created by means of combination of layers with different material characteristics – these are the so-called multiparameter phenomenological models (Jemielita, 1994).

The Vlasov elastic foundation model is a structural model. Denoting by  $q(x)$  and  $w_s(x, z)$  a load acting on the ground and a vertical displacement of the ground, respectively, one can write the equation of the two-parameter Vlasov foundation as (Vlasov & Leontiev, 1960):

$$q(x) = kw_s(x) - 2t \frac{d^2}{dx^2} w_s(x) \quad (1)$$

where:

- $k$  – stiffness coefficient of the foundation, which characterizes the compressive work of the foundation [ $\text{N} \cdot \text{m}^{-3}$ ],
- $t$  – stiffness coefficient of the foundation, which characterizes the shear work of the foundation [ $\text{N} \cdot \text{m}^{-1}$ ].

The elastic constants  $k$  and  $t$  can be determined from the formulas (Vlasov & Leontiev, 1960):

$$k = \frac{E_0 s_{11}}{1 - \nu_0^2}$$

$$t = \frac{E_0 r_{11}}{4(1 + \nu_0)}$$

where:

$$E_0 = \frac{E_s}{1 - \nu_s^2}$$

$$\nu_0 = \frac{\nu_s}{1 - \nu_s}$$

$$r_{11} = \int_0^H \vartheta^2(z) dz$$

$$s_{11} = \int_0^H \left( \frac{d\vartheta(z)}{dz} \right)^2 dz$$

where:

$H$  – thickness of a subsoil [m],

$\nu_s$  – Poisson ratio of the soil [-],

$E_s$  – Young modulus of the soil [ $\text{N} \cdot \text{m}^{-2}$ ],

The function  $\vartheta$  is the displacement disappearance function in the ground, it is assumed  $\vartheta(0) = 1$ ,  $\vartheta(H) = 0$ .

In the monograph of Vlasov and Leontiev (1960), the following functions  $\vartheta(z)$  of the disappearance of displacements along depth were proposed:

$$\vartheta(z) = 1 - \frac{z}{H} \quad (2a)$$

$$\vartheta(z) = \frac{\text{sh}[\gamma(H-z)]}{\text{sh}[\gamma H]} \quad (2b)$$

$$\vartheta(z) = e^{-\gamma z} \quad (2c)$$

where  $\gamma$  is the displacement disappearance rate along depth [ $\text{N} \cdot \text{m}^{-1}$ ].

## DIFFERENTIAL EQUATION OF A BEAM RESTING ON VLASOV FOUNDATION

Assume that the contact between the beam and the soil always exists. It means that it is satisfied an equality  $w(x) = w_s(x, 0)$ . Under this assumption, there is always an interaction between the beam and the foundation. The load acting on the beam  $p_3(x)$  is equal to

$$p_3(x) = p(x) - q(x)$$

where  $p(x)$  is external load [ $\text{N}\cdot\text{m}^{-2}$ ].

The differential equation of deflection of a beam resting on the Vlasov foundation, i.e. Eq. (1), in the Cartesian coordinate system may be written as (Vlasov & Leontiev, 1960):

$$E_b J \nabla^4 w(x) - 2t \frac{d^2}{dx^2} w(x) + k w(x) = p(x) \quad (3)$$

where:

$E_b$  – Young modulus of the beam [ $\text{N}\cdot\text{m}^{-2}$ ].

$J$  – moment of inertia of the beam [ $\text{m}^4$ ].

## APPROXIMATE SOLUTION OF A BEAM ON VLASOV SUBSOIL WITH ADDITIONAL LOAD NEAR ONE END OF THE BEAM

Consider a beam with rectangular cross-section resting freely on a single-layer elastic foundation as shown in Figure 1. Additionally, the load  $G$  is applied to the subsoil at a distance  $a$  from the right end of the beam.

The deflection of the surface layer of the subsoil in Section I for  $x \in (-\infty, -b]$ , satisfying the differential Eq. (1), can be written as:

$$w_1(x) = D_1 e^{\alpha(x+b)}$$

The solution of the differential equation of beam deflection on the Vlasov elastic foundation, i.e. Eq. (3) in Section II for  $x \in (-b, b]$  in the case of  $p(x) = \text{const}$ , can be written as:

$$w_2(x) = C_1 e^{x\theta_1} + C_2 e^{-x\theta_1} + C_3 e^{x\theta_2} + C_4 e^{-x\theta_2} + \frac{p}{k}$$

The deflection of the surface layer of the subsoil in Section III for  $x \in [b, b+a]$ , satisfying the differential Eq. (1), can be written as:

$$w_3(x) = D_2 e^{-\alpha(x-b)} + D_3 e^{\alpha(x-b)}$$

Finally, the deflection of the surface layer of the subsoil in Section IV for  $x \in [b+a, +\infty)$ , satisfying the differential Eq. (1), can be written as:

$$w_4(x) = D_4 e^{-\alpha(x-(b+a))}$$

In Eqs. (1)–(4)  $C_1 - C_4$  and  $D_1 - D_4$  are constant coefficients in units of length, and

$$\theta_1 = \sqrt{\frac{t}{E_b J} - \frac{\sqrt{t^2 - k E_b J}}{E_b J}}$$

$$\theta_2 = \sqrt{\frac{t}{E_b J} + \frac{\sqrt{t^2 - k E_b J}}{E_b J}}$$

$$\alpha = \sqrt{\frac{k}{2t}}$$

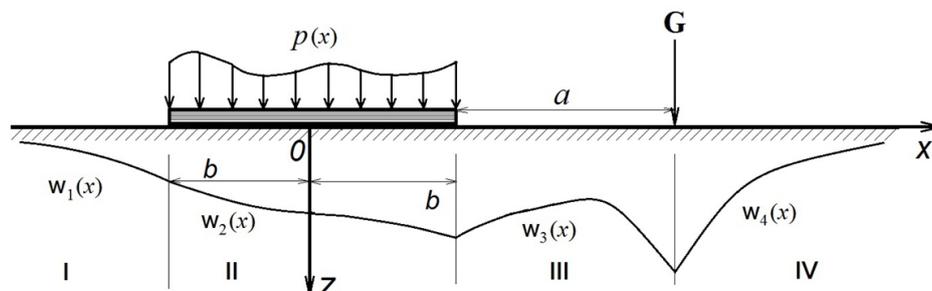


Fig. 1. Beam resting freely on an elastic foundation, with an additional load  $G$

The deflection  $w_i(x)$  corresponds to the generalized transverse force  $S_i(x)$  in the Vlasov subsoil:

$$S_i(x) = 2tw'_i(x)$$

where  $i = 1, 2, 3, 4$ .

The bending moment and the shear force in the beam are described as follows:

$$M(x) = -E_b J w_2''(x)$$

$$Q(x) = -E_b J w_2'''(x)$$

$$T(x) = S_2(x) + Q(x)$$

where  $T(x)$  is generalized cross-section force.

In order to determine the coefficients  $C_i$  and  $D_i$  the boundary conditions must be written in a form:

$$\begin{cases} w_1(-b) = w_2(-b) \\ w_2(b) = w_3(b) \\ w_3(b+a) = w_4(b+a) \\ S_3(b+a) + G = S_4(b+a) \\ M(-b) = 0 \\ M(b) = 0 \\ Q(-b) + S_2(-b) = S_1(-b) \\ Q(b) + S_2(b) = S_3(b) \end{cases}$$

### CALCULATION EXAMPLE

As an example, consider a beam with a rectangular cross-section shown in Figure 1. Assume the following geometrical dimensions and stiffness characteristics of the beam and soil:

$$b = 5 \text{ m}$$

$$H = 10 \text{ m}$$

$$\nu_s = 0.25$$

$$E_s = 50 \cdot 10^3 \text{ kN} \cdot \text{m}^{-2}$$

$$E_b = 27 \cdot 10^6 \text{ kN} \cdot \text{m}^{-2}$$

$$b_1 = 0.4 \text{ m}$$

$$h_1 = 0.5 \text{ m}$$

$$p = 10 \text{ kN} \cdot \text{m}^{-1}$$

where:

$b_1$  – beam width [m],

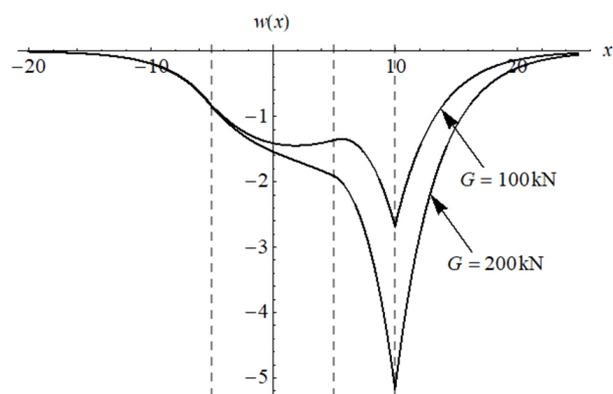
$h_1$  – beam height [m].

For a beam with rectangular cross-section, the second area moment is calculated from the formula wzor

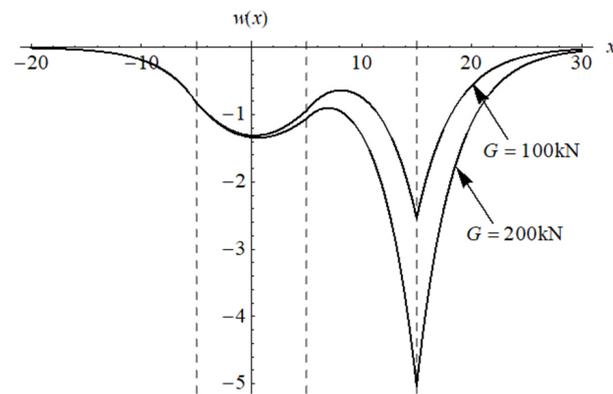
$$J = \frac{b_1 h_1^3}{12} \text{ m}^4.$$

The function of displacement disappearance along depth is assumed in a form (2a).

Figures from 2 to 9 show the diagrams of the deflection of the beam and the surface layer of the soil, as well as bending moments and transverse forces depending on the location of the additional load  $G$ .

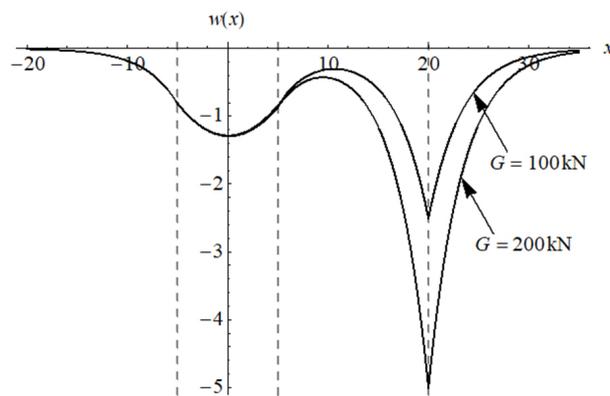


**Fig. 2.** Deflection  $w(x)$  of the beam on the Vlasov elastic foundation for  $a = b$

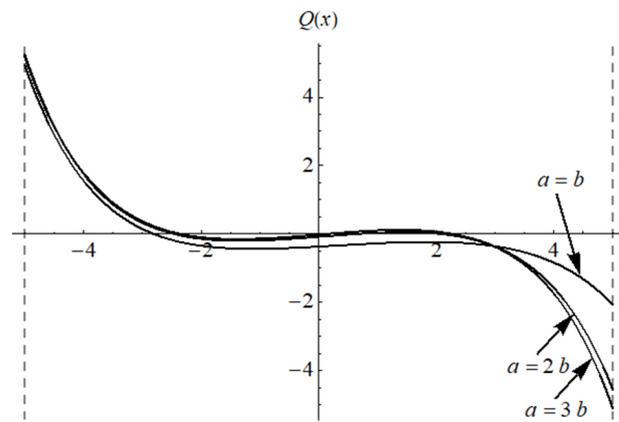


**Fig. 3.** Deflection  $w(x)$  of the beam on the Vlasov elastic foundation for  $a = 2b$

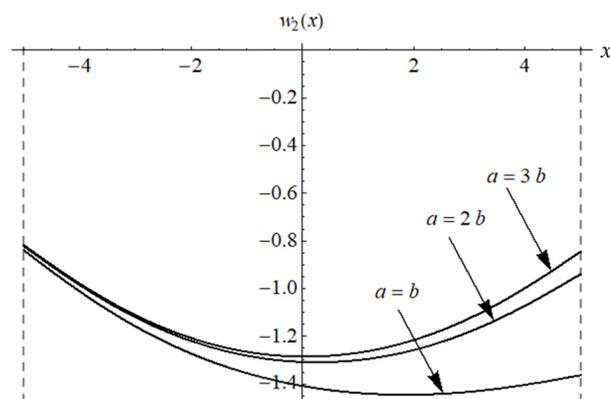
Figures 2, 3 and 4 show the effect of the distance of the additional load on the deflection of the beam for two values of the additional load  $G$ :  $G = 100 \text{ kN}$  and  $G = 200 \text{ kN}$ . Larger values of  $G$  evoke greater deflections of the beam and the subsoil surface layer. As the distance  $a$  between the additional load and



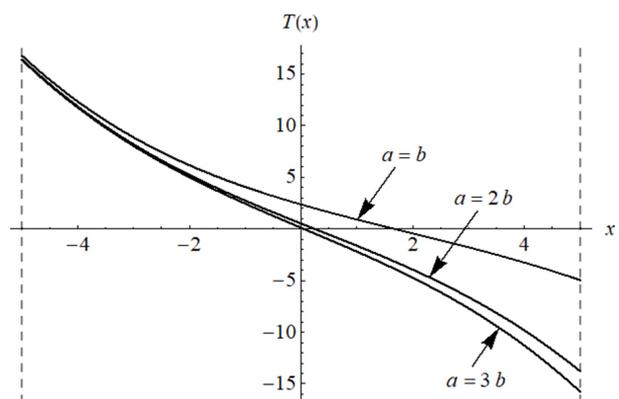
**Fig. 4.** Deflection  $w(x)$  of the beam on the Vlasov elastic foundation for  $a = 3b$



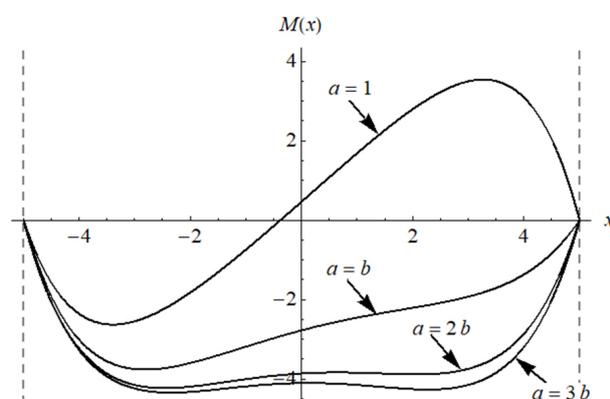
**Fig. 7.** Transverse force  $Q(x)$  in the beam depending on the position of the additional load  $G = 100$  kN



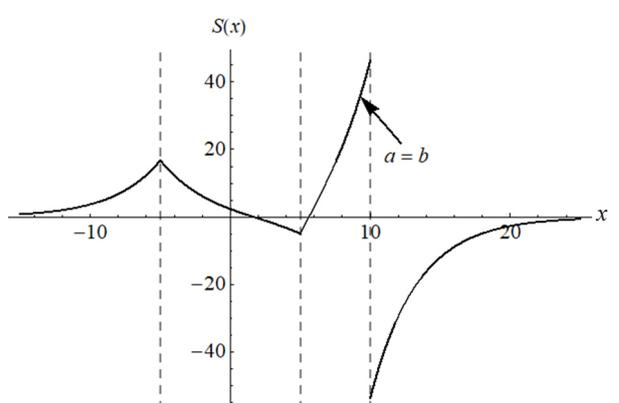
**Fig. 5.** Beam deflection  $w_2(x)$  depending on the position of the additional load  $G = 100$  kN



**Fig. 8.** Generalized transverse force  $T(x)$  of the beam depending on the position of the additional load  $G = 100$  kN



**Fig. 6.** Bending moment  $M(x)$  in the beam depending on the position of the additional load  $G = 100$  kN



**Fig. 9.** Generalized transverse forces  $S(x)$  at  $G = 100$  kN

the beam increases, the influence of this load on the beam deflection reduces. At the distance of  $a = 3b$  the influence of the load  $G$  on the beam deflection is negligible, the beam practically does not “see” this load.

Figure 5 shows the beam deflection diagram  $w_2(x)$  depending on the distance  $a$  between the beam and the point of application of the load  $G = 100$  kN.

Figures 6, 7 and 8 show the diagrams of bending moments  $M(x)$ , transverse forces  $Q(x)$  and generalized transverse forces  $T(x)$  of the beam depending on the distance  $a$  between the beam and the point of application of the load  $G = 100$  kN. If the distance  $a$  reduces, then the radius of curvature reduces accordingly what results in an increase in the influence of the load  $G$  on the values of the beam bending moments. As the value of the load  $G$  increases or the distance  $a$  decreases, the influence of the additional load on the values of the beam bending moments increases, correspondingly the symmetry of the bending moment diagram disappears (Fig. 6). Since the boundary conditions at the ends of the beam are satisfied in an exact way, the bending moments at points  $x = \pm b$  are zero, what corresponds to a case of a beam resting freely on an elastic foundation.

Figure 9 shows the generalized transverse force diagram  $S(x)$  in the subsoil (outside the beam boundaries) and the generalized transverse force in the beam  $T(x)$  for  $G = 100$  kN. At the point of application of the additional load  $G$ , the value of the generalized transverse force is equal to the value of this load. The generalized transverse force  $S(x)$  in the subsoil quickly disappears as the load is being moved away from the load application point.

## CONCLUSIONS

The paper considers the problem of the bending of a beam, resting freely on the Vlasov foundation, uniformly loaded along its entire length and being under the influence of an additional load  $G$ , applied at a distance  $a$  from the end of the beam. The influence of the additional load on the deflection and the cross-sectional forces in the beam, depending on its size and the distance between the point of application and the beam end, have been investigated. A flat model of the Vlasov subsoil was adopted, the properties of

which depend on two integral characteristics  $k$  and  $t$ , characterizing the work of the subsoil under compression and shear, respectively. The displacement disappearance function  $\vartheta(z)$ , due to the assumed thickness of the subsoil layer  $H$ , was adopted in the form (2a).

Graphs of deflection, bending moment and transverse force of the beam were prepared depending on the distance and position of an additional load  $G$ . In the case of a very small distance  $a$ , the bending moments change the sign on the right half of the beam. This effect should be taken into account when calculating the effect of the additional load on already existing foundations. The closer the additional load is to the beam end, the greater its influence on the beam deflection and cross-sectional forces.

## Authors' contributions

Conceptualization: M.N.; methodology: M.N.; validation: M.N. and K.R.; formal analysis: M.N. and K.R.; investigation: M.N. and K.R.; resources: M.N.; data curation: M.N. and K.R.; writing – original draft preparation: M.N.; writing – review and editing: M.N. and K.R.; visualization: M.N. and K.R.; supervision: M.N. and K.R.; project administration: M.N. and K.R.; funding acquisition: K.R. and M.N.

All authors have read and agreed to the published version of the manuscript.

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## **BELKA SPRĘŻYSTA SPOCZYWAJĄCA NA PODŁOŻU SPRĘŻYSTYM WŁASOWA O WYBRANYCH WARUNKACH BRZEGOWYCH**

### **STRESZCZENIE**

W pracy przedstawiono zagadnienie zginania belki obciążonej równomiernie na całej długości, swobodnie spoczywającej na sprężystym podłożu Własowa z dodatkowym obciążeniem zewnętrznym  $G$  podłoża przyłożonym w odległości  $a$  od końca belki. Podany przykład jest szczególnym przypadkiem belki swobodnie spoczywającej na sprężystym podłożu, występującej w praktyce budowlanej w wielu przypadkach. W pracy rozpatrzono przybliżone rozwiązanie wpływu dodatkowego obciążenia  $G$  na ugięcie oraz siły przekrojowe belki swobodnie spoczywającej na sprężystym podłożu gruntowym typu Własowa. Przedstawiono wykresy ugięcia belki oraz warstwy powierzchniowej gruntu poza jej granicami, a także wykresy momentów zginających i sił poprzecznych w belce. Zbadano wpływ odległości  $a$  przyłożenia dodatkowego obciążenia  $G$  na ugięcie oraz siły przekrojowe w belce.

**Słowa kluczowe:** belka sprężysta, podłoże sprężyste Własowa, siły przekrojowe, ugięcie belki, funkcja zanikania przemieszczeń