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ORIGINAL PAPER

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STATIC ANALYSIS OF A VARIABLE CROSS-SECTION ELEMENT UNDER LOAD

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ABSTRACT

An incremental solution of a member in bending and compression by displacement method with shear force impact is presented, wherein individual incremental steps, both the geometrical characteristics of the cross-section and the position of the neutral axis, can be changed. Functions of loads, internal forces and displacements are represented by trigonometric series. In every load step internal forces and deformation state is memorised, therefore history of the load and stiffness is taken into account. This solution is useful both in static calculations of structures being strengthened as well as in structures, in which degradation of stiffness under load occurs as a result of wall stability loss. The member may be loaded by uniformly distributed load, any number of concentrated forces and nodal moments.

Key words: incremental solution by displacement method, strengthening of structures, degradation of stiffness

INTRODUCTION

Strengthening of steel bar structures is performed in the presence of displacement and internal forces states, existing at the start of construction work. It may consist in enlargement of the member cross-section along its entire length or its part, or introducing additional restraints in any cross-section or in structure nodes. Therefore, to assess the effort of the strengthened member, an incremental solution with possibility of changing the stiffness of the member at every calculation step is needed. Such an approach allows the analysis of structures strengthened by symmetrical enlargement of the cross-section relative to the neutral axis. If change of the cross-section neutral axis position occurs, it is still necessary to take into account the effect of nodal bending moments resulting from the axial force and the eccentric determined by the axis movement.

In turn, if analysed structure is strengthened by introducing additional restraints along the length of the member or at any node, it is necessary to consider such a possibility in the computer program algorithm by introducing into calculations the final configuration of nodes, rods and restraints like for non-deformed structure and the parameter controlling the rigidity of the restraint and its initial deformation. For example, introducing an additional support of initially and elastically deformed member would require introducing in any incremental step a movement of this additional restraint relative to the non-deformed structure by the sum of the values of initial and elastic member deformation. In addition, the current sum of initial and elastic deflection of two members formed as a result of supporting the original member, would have to be corrected. Similar operation would have to be carried out in case of change in the cross-section of analysed member along part of its length, with introduction of two additional nodes along the length of such a member, in the initial and in the end sections of the strengthened segment.

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A similar solution would be needed in the event of degradation of the bar stiffness due to stability loss of its slender wall exposed to normal stress. However, due to the possible change in the value of normal stress along the bar length, such a solution should be treated as approximate as an effect of the need of splitting the bar into short elements, for which the stress values can be considered approximately constant along their lengths. In such elements, in subsequent load increase steps, the surface area, moment of inertia and position of the neutral axis would change.

There are solutions and calculation software, in which the impact of local instability of slender walls on the member global deformations is considered. In general, these programs use shell models of bar (Dassault Systèmes, 2013; ERGOCAD, 2020) and can be used to analyse individual thin-walled members or simple frames (Giżejowski, Szczerba, Gajewski & Stachura, 2015). These models are accurate but their implementation is time-consuming. In PhD dissertation (Czepiżak, 2006) the effect of trapezoidal sheets stiffness degradation due to the local instability of their walls is captured by function of vanishing stiffness $r_e(M)$, describing the change in the stiffness of the corrugated sheet cross-section as a function of utilization its bending moment resistance. This solution can be used in the first-order analysis of multi-span members subjected to bending without axial force. There are also solutions take into account degradation of the member stiffness due to the development of the plastic zone associated with the appearance of plastic hinges (Barszcz & Giżejowski, 2006). Unfortunately, these solutions are of little use to the engineer. They can only be used in individual cases to a limited extent.

The article presents formulas used in displacement method in an incremental approach for initially deformed bar, in compression and bending in the *xy* plane. These formulas may be used to assess the load capacity of members being strengthened, or whose cross-section is degraded due to loss of wall stability. Similar formulas can be derived for bending in the *xz* plane, assuming the displacements and strains are low enough to make it possible to sum the stress caused by the impact of loads acting in both main planes, at the bar curvature determined from two independent formulas.Such a solution can be still supplemented with the impact of node flexibility (Zamorowski, 2013).

BENDING IN THE *xy* PLANE

Division of the load into any number of incremental steps is introduced. In each incremental step, it is possible to change the cross-section, assuming this change will not affect the existing strain of the bar. Such a model of the bar allow the analysis of the structure with sequential application of loads (one after another), including alternating loads, as well as the analysis of erected or strengthened structure with variable topology. Two local coordinate systems were introduced – in the $\overline{p}$ and $\overline{k}$ node with positive senses as in Figure 1.

It was assumed that the positive senses of initial and elastic strain of the bar and loads along its length are consistent with the positive senses of the system axis in the node $\overline{p}$, whereas positive senses of nodal loads, nodal forces and node displacements correspond to the positive senses of the axes of the local systems starting at these nodes.

It is assumed that in any incremental step k the bending rigidity of the bar is equal to $EJ_{z,k}$, and shear rigidity equal to GA_k. It is adopted that the bar deflection is the sum of initial bending ($u_{y,0}$), deflection caused by bending impact ($u_{y,M}$) and deflection caused by shear force ($u_{y,V}$) – see Figure 2.

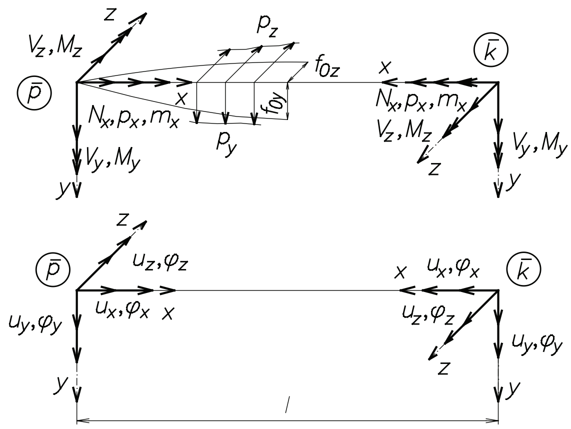

Fig. 1. Designation of load components, displacements and nodal forces

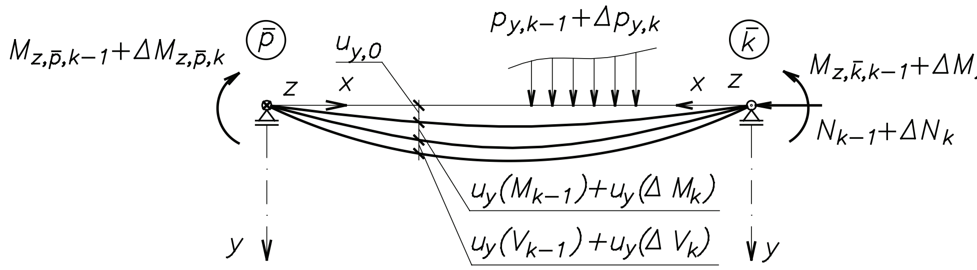

Fig. 2. Model of a to-pinned bar bent in the *xy* plane

Total deflection in the first incremental step is

$$\Delta u_{y,1} = u_{y,0} + \Delta u_{y,M,1} + \Delta u_{y,V,1} \tag{1}$$

Increase in bending moment and shear force over the length of the bar is determined by the following equations

$$\begin{gathered}\Delta M_{z,1}(x) = \Delta M_{z,1}(p) + \Delta N_1 \cdot \Delta u_{y,1} \\ \Delta V_{y,1}(x) = \Delta V_{y,1}(p) + \Delta N_1 \cdot \frac{d\Delta u_{y,1}}{dx}\end{gathered} \tag{2a}$$

where

$$\begin{gathered}\Delta M_{z,1}(p) = \Delta M_{z,1}(M_{z,\bar{p}}) + \Delta M_{z,1}(p_y) + \\ + \Delta M_{z,1}(P_{y,i}) + \Delta M_{z,1}(M_{z,\bar{k}}) \\ \Delta V_{y,1}(p) = \Delta V_{y,1}(M_{z,\bar{p}}) + \Delta V_{y,1}(p_y) + \\ + \Delta V_{y,1}(P_{y,i}) + \Delta V_{y,1}(M_{z,\bar{k}})\end{gathered} \tag{2b}$$

Angle of shear deformation

$$\begin{gathered}\frac{d\Delta u_{y,V,1}}{dx} = a_{y,1}\frac{\Delta V_{y,1}(x)}{GA_1} = \\ = \gamma_{y,1}\left[\Delta V_{y,1}(p) + \Delta N_1 \frac{d\Delta u_{y,1}}{dx}\right]\end{gathered} \tag{3}$$

where (Belyayev, 1956)

$$a_{y,1} = \frac{A_1}{J_{z,1}^2}\int_{-h_{1,1}}^{h_{2,1}} \frac{S_{z,1}^2(y)}{b_1(y)}dy \approx \frac{A_1}{A_{v,1}},$$

in which

$b_1(y)$ and $h_1 = h_{1,1} + h_{2,1}$ – the section width and height in the first step,
A_1, $A_{v,1}$ – cross-sectional area, shear cross-sectional area in the first step,
$J_{z,1}$, $S_{z1}(y)$ – moment of inertia of the cross-section, static moment of the cut-off part of the cross-section.

The equation for the curvature of the bar in bending assuming small displacements is adopted in the following equation

$$\frac{1}{\rho_y(x)} \cong \frac{d^2\Delta u_{y,M,1}}{dx^2} = -\frac{\Delta M_{z,1}(x)}{EJ_{z,1}} \tag{4}$$

By applying of double differentiation of Eq. (1) is obtained

$$\frac{d^2\Delta u_{y,1}}{dx^2} = \frac{d^2 u_{y,0}}{dx^2} + \frac{d^2\Delta u_{y,M,1}}{dx^2} + \frac{d^2\Delta u_{y,V,1}}{dx^2} \tag{5}$$

and, after introducing Eq. (2a) to Eq. (4) and taking into account the derivative (5) and after ordering, it is obtained

$$\begin{gathered}\frac{d^2\Delta u_{y,1}}{dx^2} + k_{y,1}^2\Delta u_{y,1} = -\frac{\eta_{y,1}}{EJ_{z,1}}\Delta M_{z,1}(p) + \\ + \eta_{y,1}\frac{d^2\Delta u_{y,0}}{dx^2} + \eta_{y,1}\gamma_{y,1}\frac{d\Delta V_{y,1}(p)}{dx}\end{gathered} \tag{6}$$

for $\eta_{y,1} = \frac{1}{(1-\gamma_{y,1}N_1)}$, $N_1 = \Delta N_1$, $k_{y,1}^2 = \frac{\eta_{y,1}N_1}{EJ_{z,1}}$

In the second incremental step

$$\Delta u_{y,2} = \Delta u_{y,M,2} + \Delta u_{y,V,2} \tag{7}$$

$$\Delta M_{z,2}(x) = \Delta M_{z,2}(p) + N_1 \Delta u_{y,2} + \Delta N_2 \left(\Delta u_{y,1} + \Delta u_{y,2}\right) = \Delta M_{z,2}(p) + N_2 \Delta u_{y,2} + \Delta N_2 \Delta u_{y,1} \tag{8a}$$

$$\text{for} \quad \Delta M_{z,2}(p) = \Delta M_{z,2}(M_{z,\bar{p}}) + \Delta M_{z,2}(p_y) + \Delta M_{z,2}(P_{y,i}) + \Delta M_{z,2}(M_{z,\bar{k}})$$

where

$$\Delta M_{z,\bar{p}} = \Delta M_{z,\bar{p},2}(p) + N_2 \Delta e_{y,2}$$
$$\Delta M_{z,\bar{k}} = \Delta M_{z,\bar{k},2}(p) + N_2 \Delta e_{y,2} \tag{8b}$$

wherein

$N_2 = \Delta N_1 + \Delta N_2$, $\Delta e_{y,2}$ – change of neutral axis position in the second step,

$\Delta M_{z,\bar{p},2}(p)$, $\Delta M_{z,\bar{k},2}(p)$ – increase in external nodal moments in nodes $\bar{p}$ and $\bar{k}$ in the second step.

Considering that

$$\Delta V_{y,2}(x) = \Delta V_{y,2}(p) + N_2 \frac{d\Delta u_{y,2}}{dx} + \Delta N_2 \frac{d\Delta u_{y,1}}{dx}$$

$$\frac{d\Delta u_{y,V,2}}{dx} = \gamma_{y,2} \Delta V_{y,2}(x) = \gamma_{y,2} \left[\Delta V_{y,2}(p) + N_2 \frac{d\Delta u_{y,2}}{dx} + \Delta N_2 \frac{d\Delta u_{y,1}}{dx}\right] \tag{9}$$

$$\gamma_{y,2} = \frac{1}{GA_{v,2}}$$

in the second step the following equation is obtained

$$\frac{d^2 \Delta u_{y,2}}{dx^2} + k_{y,2}^2 \Delta u_{y,2} = -\frac{\eta_{y,2}}{EJ_{z,2}}\left[\Delta M_{z,2}(p) + \Delta N_2 \Delta u_{y,1}\right] + \gamma_{y,2}\eta_{y,2}\left[\frac{d\Delta V_{y,2}(p)}{dx} + \Delta N_2 \frac{d^2 \Delta u_{y,1}}{dx^2}\right] \tag{10}$$

where $\eta_{y,2} = \dfrac{1}{1 - \gamma_{y,2} N_2}$ and $k_{y,2}^2 = \dfrac{\eta_{y,2} N_2}{EJ_{z,2}}$.

Similarly, for incremental step k (with $k > 1$), the following dependencies are obtained:

$$\Delta u_{y,k} = \Delta u_{y,M,k} + \Delta u_{y,V,k} \tag{11}$$

$$\Delta M_{z,k}(x) = \Delta M_{z,k}(p) + N_k \Delta u_{y,k} + \Delta N_k \sum_{m=1}^{k-1} \Delta u_{y,m} \tag{12a}$$

$$\text{for } \Delta M_{z,2}(p) = \Delta M_{z,2}(M_{z,\bar{p}}) + \Delta M_{z,2}(p_y) + \Delta M_{z,2}(P_{y,i}) + \Delta M_{z,2}(M_{z,\bar{k}})$$

where

$$\Delta M_{z,\bar{p}} = \Delta M_{z,\bar{p}}(p) + N_k \Delta e_{y,k} + \Delta N_k \sum_{m=2}^{k-1} \Delta e_{y,m}$$
$$\Delta M_{z,\bar{k}} = \Delta M_{z,\bar{k}}(p) + N_k \Delta e_{y,k} + \Delta N_k \sum_{m=2}^{k-1} \Delta e_{y,m} \tag{12b}$$

wherein

$N_k = \sum_{m=1}^{k} \Delta N_m$, $\Delta e_{y,2}$ – change the position of the neutral axis in k-th step,

$\Delta M_{z,\bar{p},k}(p)$, $\Delta M_{z,\bar{k},k}(p)$ – increase in external node moments in nodes $\bar{p}$ and $\bar{k}$ in the k-th step.

Considering that

$$\Delta V_{y,k}(x) = \Delta V_{y,k}(p) + N_k \frac{d\Delta u_{y,k}}{dx} + \Delta N_k \sum_{m=1}^{k-1} \frac{d\Delta u_{y,m}}{dx} \tag{13}$$

differential equation of deformed axis in incremental step *k* takes the following form

$$\frac{d^2 \Delta u_{y,k}}{dx^2} + k_{y,k}^2 \Delta u_{y,k} = -\frac{\eta_{y,k}}{EJ_{z,k}} \left[\Delta M_{z,k}(p) + \Delta N_k \sum_{m=1}^{k-1} \Delta u_{y,m} \right] + \gamma_{y,k} \eta_{y,k} \left[\frac{d \Delta V_{y,k}(p)}{dx} + \Delta N_k \sum_{m=1}^{k-1} \frac{d^2 \Delta u_{y,m}}{dx^2} \right] \tag{14}$$

where: $\eta_{y,k} = \frac{1}{1 - \gamma_{y,k} N_k}$, $k_{y,k}^2 = \frac{\eta_{y,k} N_k}{EJ_{z,k}}$ and $\gamma_{y,k} = \frac{a_{y,k}}{(GA_k)}$ $a_{y,k} = \frac{A_k}{J_{z,k}^2} \int_{-h_1}^{h_2} \frac{S_{z,k}^2(y)}{b_k(y)} dy$.

The functions of initial bending of the bar ($u_{y,0}$), elastic bending increments ($\Delta u_{y,k}$), internal bending moments increments ($\Delta M_{z,k}$) and load increments ($\Delta p_{y,k}$) are assumed to be infinite trigonometric series

$$\begin{aligned} u_{y,0} &= \sum_n f_{y,0,n} \sin \frac{n\pi x}{l} \\ \Delta u_{y,k} &= \sum_n f_{y,k,n} \sin \frac{n\pi x}{l} \\ \Delta M_{z,k}(p) &= \sum_n \Delta m_{z,k,n} \sin \frac{n\pi x}{l} \\ \Delta p_{y,k} &= \sum_n \Delta p_{y,k,n} \sin \frac{n\pi x}{l} \\ n &= 1, \ 2, \ 3, \ldots \end{aligned} \tag{15}$$

The amplitudes of the increments of bending moments from external loads were obtained from the equation

$$\frac{d^2 \Delta M_{z,k}(p)}{dx^2} = \frac{d \Delta V_{y,k}(p)}{dx} = -\Delta p_{y,k} \tag{16a}$$

in which $\Delta p_{y,k}$ is the *k*-th increment of transverse load.

The following equation was obtained

$$\Delta m_{z,k,n} = \left(\frac{l}{n\pi} \right)^2 \Delta p_{y,k,n} \tag{16b}$$

where (acc. to Girkmann, 1956) amplitude for an uniformly distributed load

$$\begin{aligned} \Delta p_{y,k,n}^p &= \frac{4 \Delta p_{y,k}}{n\pi} \sin^2 \frac{n\pi}{2} \\ \Delta m_{z,p,k,n} &= \frac{4 \Delta p_{y,k} l^2}{(n\pi)^3} \sin^2 \frac{n\pi}{2} \end{aligned} \tag{16c}$$

In case of concentrated load of simple-supported beam it can be written

$$\begin{aligned} \Delta p_{y,k,n}^P &= \frac{2 \Delta P_{y,i,k}}{l} \sin \frac{n\pi a_i}{l} \\ \Delta m_{z,P,k,n} &= \frac{2 \Delta P_{y,i,k} l}{(n\pi)^2} \sin \frac{n\pi a_i}{l} \end{aligned} \tag{16d}$$

where a_i is the distance of the force $P_{y,i}$ from the node $\overline{p}$.

The series for concentrated force is not convergent, but for the bending moment series is. Assuming the concentrated force in the middle of the bar span, differences in the values of the maximum bending moment are about 1% with n = 40 and 0.5% with n = 80 and 0.25% with n = 160 compared to precise solution.

Amplitudes of bending moments from loading with moments in the node $\overline{p}$ and $\overline{k}$

$$\Delta m_{z,\overline{p},k,n} = \frac{2\Delta M_{z,\overline{p},k}}{n\pi}$$
$$\Delta m_{z,\overline{k},k,n} = -\frac{2\Delta M_{z,\overline{k},k}}{n\pi}\cos n\pi \tag{16e}$$

In case of a beam loaded with bending moments in the support nodes, the differences between the value of bending moments obtained according to the first order theory and calculated from Eq. (15) with the amplitudes according to Eq. (16e), are larger in cross-sections close to the point of application of the load and smaller in the middle of the beam span. Assuming, for example, n = 1,000, these differences are: 2% in a cross-section at a distance of $0.01l$ from the node, 0.2% in cross-section at a distance of $0.1l$ and 0.06% in the middle of the bar span.

Substituting e.g. Eqs. (16c) and (16e) to Eq. (15) for a uniformly distributed load and moments in nodes $\overline{p}$ and $\overline{k}$ it is obtained

$$\Delta M_{z,k}(p) = \sum_n \left[\frac{4\Delta p_{y,k} l^2}{(n\pi)^3}\cdot \sin^2\frac{n\pi}{2} + \frac{2\Delta M_{z,p,k}}{n\pi} - \frac{2\Delta M_{z,\overline{k},k}}{n\pi}\cos n\pi\right]\sin\frac{n\pi x}{l} \tag{17}$$

Assuming designations as in Eqs. (16c) and (16e)

$$\frac{d\Delta V_{y,k}(p)}{dx} = \frac{d^2\Delta M_{z,k}(p)}{dx^2} = -\sum_n \left(\frac{n\pi}{l}\right)\left[\Delta m_{z,p,k,n} + \Delta m_{z,\overline{p},k,n} + \Delta m_{z,\overline{k},k,n}\right]\sin\frac{n\pi x}{l} \tag{18}$$

Substituting Eqs. (15), (17) and (18) and derivatives of displacements (15) to (6), the elastic curve equation describing deflection of the bar with the uniformly distributed load over the length and with moments in nodal cross-sections is obtained.

In the first incremental step

$$-\sum_n \left(\frac{n\pi}{l}\right)^2 f_{y,1,n}\sin\frac{n\pi x}{l} + k_{y,1}^2 \sum_n f_{y,1,n}\sin\frac{n\pi x}{l} = -\frac{\eta_{y,1}}{EJ_{z,1}}\sum_n\left[\Delta m_{z,p,1,n} + \Delta m_{z,\overline{p},1,n} + \Delta m_{z,\overline{k},1,n}\right]\times$$
$$\times\sin\frac{n\pi x}{l} - \eta_{y,1}\sum_n\left(\frac{n\pi}{l}\right)^2 f_{y,0,n}\sin\frac{n\pi x}{l} - \eta_{y,1}\gamma_{y,1}\sum_n\left(\frac{n\pi}{l}\right)^2\left[\Delta m_{z,p,1,n} + \Delta m_{z,\overline{p},1,n} + \Delta m_{z,\overline{k},1,n}\right]\sin\frac{n\pi x}{l} \tag{19}$$

Hence, the bar bending amplitude after adding the influence of the concentrated force according to Eq. (16d), for any harmonic component n in the first incremental step is

$$f_{y,1,n} = \frac{\Delta m_{z,p,1,n} + \Delta m_{z,P,1,n} + \Delta m_{z,\overline{p},1,n} + \Delta m_{z,\overline{k},1,n}}{\dfrac{N_{kr,z,1,n}}{1+\gamma_{y,1}N_{kr,z,1,n}} - N_1} + \frac{\eta_{y,1}\left(\dfrac{n\pi}{l}\right)^2 f_{y,0,n}}{\left(\dfrac{n\pi}{l}\right)^2 - k_{y,1}^2} \tag{20a}$$

where $N_{kr,z,1,n} = \dfrac{n^2\pi^2 EJ_{z,1}}{l^2}$.

Similarly, for the incremental step k:

$$f_{y,k,n} = \frac{\Delta m_{z,p,k,n} + \Delta m_{z,\bar{p},k,n} + \Delta m_{z,\bar{k},k,n} + \Delta N_k \sum_{m=1}^{k-1} f_{y,m,n}}{\dfrac{N_{kr,z,k,n}}{1+\gamma_{y,k}\, N_{kr,z,k,n}} - N_k} \tag{20b}$$

$$N_{kr,z,k,n} = \frac{n^2 \pi^2 EJ_{z,k}}{l^2}$$

where $\sum_{m=1}^{k-1} f_{y,m,n}$ is the sum of the amplitudes of elastic displacements from previous incremental steps.

Substituting the amplitudes (20a) and (20b) in Eq. (15), it is possible to determine the elastic deflection of the bar $\Delta u_{y,k}$ in any cross-section loaded with concentrated forces and the uniform load distributed over its length and moments in node cross-sections.

The increase in the elastic angle of rotation from bar bending in the first and in the k-th incremental steps is calculated from the following formulas (Fig. 3):

$$\phi_{z,1}(x) = \frac{du_{y,1}}{dx} - \frac{du_{y,V,1}}{dx} - \frac{du_{y,0}}{dx} = \frac{du_{y,1}}{dx} - \gamma_{y,1}\Delta V_{y,1}(x) - \frac{du_{y,0}}{dx}$$

$$\phi_{z,k}(x) = \frac{du_{y,k}}{dx} - \frac{du_{y,V,k}}{dx} = \frac{du_{y,k}}{dx} - \gamma_{y,k}\Delta V_{y,k}(x) \tag{21}$$

In the first step, for increase of evenly distributed load with increase in moments $M_{z,\bar{p},1}$, $M_{z,\bar{k},1}$ is obtained

$$\Delta\varphi_{z,1}(x) = \frac{d\Delta u_{y,1}}{dx} - \gamma_{y,1}\left[\Delta V_{1y},(p) + \Delta N_1 \frac{d\Delta u_{y,1}}{dx}\right] - \frac{du_{y,0}}{dx} = \frac{d\Delta u_{y,1}}{dx}\left(1-\gamma_{y,1}\Delta N_1\right) -$$

$$-\gamma_{y,1}\Delta V_{y,1}(p) - \frac{du_{y,0}}{dx} = \frac{d\Delta u_{y,1}}{dx}\left(1-\gamma_{y,1}\Delta N_1\right) - \gamma_{y,1}\frac{d\Delta M_{z,1}(p)}{dx} - \frac{du_{y,0}}{dx} \tag{22a}$$

and, taking into account derivatives of displacements and moments

$$\Delta\varphi_{z,1}(x) = -\gamma_{y,1}\sum_n \frac{n\pi}{l}\left[\Delta m_{z,p,k,n} + \Delta m_{z,P,1,n} + \Delta m_{z,\bar{p},1,n} + \Delta m_{z,\bar{k},1,n}\right]\times$$

$$\times\cos\frac{n\pi x}{l} + \left(1-\gamma_{y1}\Delta N_1\right)\sum_n \frac{n\pi}{l} f_{y,1,n}\cos\frac{n\pi x}{l} - \sum_n \frac{n\pi}{l} f_{y,0,n}\cos\frac{n\pi x}{l} \tag{22b}$$

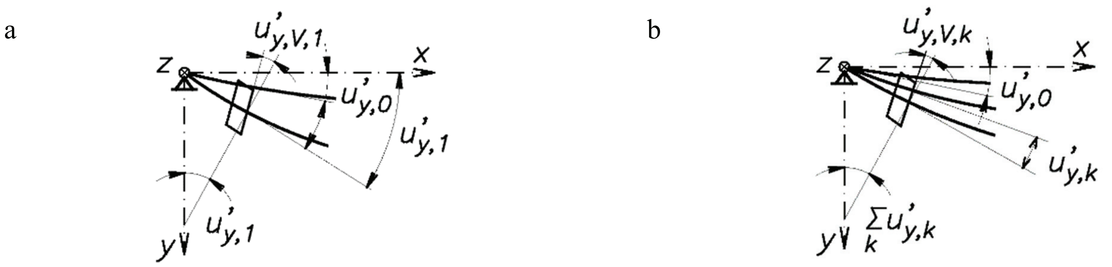

Fig. 3. Angular displacements of the cross-section near the joint in the *xy* plane: a – at the first step of increment; b – at the *k*-th step

In the incremental step k:

$$\Delta\varphi_{z,k}(x) = \frac{d\Delta u_{y,k}}{dx}\left(1-\gamma_{y,k}N_k\right) - \gamma_{y,k}\left[\frac{d\Delta M_{z,k}(p)}{dx} + \Delta N_k \sum_{m=1}^{k-1}\frac{d\Delta u_{y,m}}{dx}\right] \tag{22c}$$

and after applying expressions for derivatives of displacements and moments

$$\begin{gathered}\Delta\varphi_{z,1}(x) = -\gamma_{y,1}\sum_n \frac{n\pi}{l}\left[\Delta m_{z,p,k,n} + \Delta m_{z,P,1,n} + \Delta m_{z,\bar{p},1,n} + \Delta m_{z,\bar{k},1,n}\right]\times \\ \times\cos\frac{n\pi x}{l} + \left(1-\gamma_{y1}\Delta N_1\right)\sum_n \frac{n\pi}{l} f_{y,1,n}\cos\frac{n\pi x}{l} - \sum_n \frac{n\pi}{l} f_{y,0,n}\cos\frac{n\pi x}{l}\end{gathered} \tag{22d}$$

In case of bar initially deformed in the xy plane, loaded with a bending moment in the node $\bar{p}$ as in Figure 4, the increase in the elastic angle of rotation in the first step is calculated according to Eq. (22b), taking into account the amplitude of the moment as in Eq. (16e).

$$\Delta\varphi_{z,1}(x) = \sum_n \frac{n\pi}{l}\left[f_{y,1,n}\left(1-\gamma_{y,1}\Delta N_1\right) - \gamma_{y,1}\frac{2\Delta M_{z,\bar{p},1}}{n\pi} - f_{y,0,n}\right]\cos\frac{n\pi x}{l} \tag{23}$$

Introducing the amplitude of the elastic bending deflection to Eq. (23), in the form resulting from Eq. (20a) for the moment load in the node $\bar{p}$, assuming $x = 0$, an expression for the increase of the rotation angle in the first incremental step was obtained

$$\Delta\varphi_{z,\bar{p},1} = \frac{\Delta M_{z,\bar{p},1}}{l}\sum_n \frac{2 + \dfrac{n\pi N_1 f_{y,0,n}}{\Delta M_{z,\bar{p},1}}}{N_{kr,z,1,n} - N_1\left(1+\gamma_{y,1}N_{kr,z,1,n}\right)} \tag{24a}$$

Introducing the designation

$$C_{z,\bar{p},1}^{\bar{p}} = \frac{3EJ_{z,1}}{l^2}\sum_n \frac{2 + \dfrac{n\pi N_1 f_{y,0,n}}{\Delta M_{z,\bar{p},1}}}{N_{kr,z,n} - N_1\left(1+\gamma_{y,1}N_{kr,z,1,n}\right)} \tag{24b}$$

in which the upper index describes the location of load application point, and the second lower index – the rotated node

$$\Delta\varphi_{z,\bar{p},1}^{\bar{p}} = \frac{\Delta M_{z,\bar{p},1}\, l}{3EJ_{z,1}} C_{z,\bar{p},1}^{\bar{p}} \tag{24c}$$

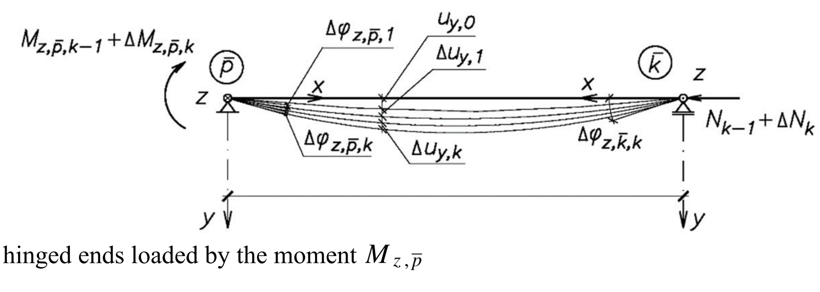

Fig. 4. A bar with hinged ends loaded by the moment $M_{z,\bar{p}}$

Similarly, for $x = l$

$$\Delta\varphi^{\bar{p}}_{z,\bar{k},1} = \frac{\Delta M_{z,\bar{p},1}\, l}{6EJ_{z,1}} C^{\bar{p}}_{z,\bar{k},1}$$

$$\text{for } C^{\bar{p}}_{z,\bar{k},1} = \frac{6EJ_{z,1}}{l^2} \sum_n \frac{2 + \dfrac{n\pi\, N_1 f_{y,0,n}}{\Delta M_{z,\bar{p},1}}}{N_{kr,z,1,n} - N_1(1+\gamma_{y,1} N_{kr,z,1,n})} \cos n\pi \tag{24d}$$

In incremental step k

$$\Delta\varphi_{z,k}(x) = \sum_n \frac{n\pi}{l}\left[f_{y,k,n}\left(1-\gamma_{y,k} N_k\right) - \gamma_{y,k}\frac{2\Delta M_{z,\bar{p},k}}{n\pi} - \gamma_{y,k}\Delta N_k \sum_{m=1}^{k-1} f_{y,m,n}\right]\cos\frac{n\pi x}{l} \tag{25a}$$

and

$$\Delta\varphi^{\bar{p}}_{z,\bar{p},k} = \frac{\Delta M_{z,\bar{p},k}\, l}{3EJ_{z,k}} C^{\bar{p}}_{z,\bar{p},k}$$

$$\Delta\varphi^{\bar{p}}_{z,\bar{k},k} = \frac{\Delta M_{z,\bar{p},k}\, l}{6EJ_{z,k}} C^{\bar{p}}_{z,\bar{k},k} \tag{25b}$$

where

$$C^{\bar{p}}_{z,\bar{p},k} = \frac{3EJ_{z,k}}{l^2} \sum_n \frac{2 + \dfrac{n\pi\Delta N_k \sum_{m=1}^{k-1} f_{y,m,n}}{\Delta M_{z,\bar{p},k}}}{N_{kr,z,k,n} - N_k\,(1+\gamma_{y,k} N_{kr,z,k,n})}$$

$$C^{\bar{p}}_{z,\bar{k},k} = \frac{6EJ_{z,k}}{l^2} \sum_n \frac{2 + \dfrac{n\pi\Delta N_k \sum_{m=1}^{k-1} f_{y,m,n}}{\Delta M_{z,\bar{p},k}}}{N_{kr,z,k,n} - N_k\,(1+\gamma_{y,k} N_{kr,z,k,n})} \cos n\pi \tag{25c}$$

For a bar loaded with a moment in the end node as in Figure 5 the increase in the elastic angle of rotation in the first step is calculated according to Eq. (22b), after taking into account the amplitude of the moment as in Eq. (16e) and the displacement amplitude as in Eq. (20a). The following equation was obtained

$$\Delta\varphi_{z,1}(x) = \frac{\Delta M_{z,\bar{k},1}}{l} \sum_n \frac{\dfrac{n\pi\, N_1 f_{y,0,n}}{\Delta M_{z,\bar{k},1}} - 2\cos n\pi}{N_{kr,z,1,n} - N_1(1+\gamma_{y,1} N_{kr,z,1,n})} \cos\frac{n\pi x}{l} \tag{26a}$$

and hence for $x = 0$ and for $x = l$

$$\Delta\varphi^{\bar{k}}_{z,\bar{p},1} = \frac{\Delta M_{z,\bar{k},1}\, l}{6EJ_{z,1}} C^{\bar{k}}_{z,\bar{p},1}$$

$$\Delta\varphi^{\bar{k}}_{z,\bar{k},1} = \frac{\Delta M_{z,\bar{k},1}\, l}{3EJ_{z,1}} C^{\bar{k}}_{z,\bar{k},1} \tag{26b}$$

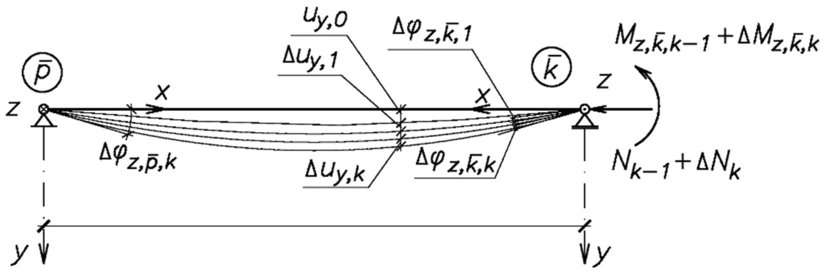

Fig. 5. A bar with hinged ends loaded by the moment $M_{z,\bar{k}}$

wherein

$$C^{\bar{k}}_{z,\bar{p},1} = \frac{6EJ_{z,1}}{l^2}\sum_n \frac{\dfrac{n\pi\, N_1 f_{y,0,n}}{\Delta M_{z,\bar{k},k}} - 2\cos n\pi}{N_{kr,z,1,n} - N_1(1+\gamma_{y,1} N_{kr,z,1,n})}$$

$$C^{\bar{k}}_{z,\bar{k},1} = \frac{3EJ_{z,1}}{l^2}\sum_n \frac{\dfrac{n\pi\, N_1 f_{y,0,n}}{\Delta M_{z,\bar{k},k}} - 2\cos n\pi}{N_{kr,z,1,n} - N_1(1+\gamma_{y,1} N_{kr,z,1,n})}\cos n\pi \tag{26c}$$

In the incremental step k

$$\Delta\varphi_{z,k}(x) = \frac{\Delta M_{z,\bar{k},k}}{l}\sum_n \frac{\dfrac{n\pi\, \Delta N_k \sum_{m=1}^{k-1} f_{y,m,n}}{\Delta M_{z,\bar{k},k}} - 2\cos n\pi}{N_{kr,z,k,n} - N_k(1+\gamma_{y,k} N_{kr,z,k,n})}\cos\frac{n\pi x}{l} \tag{27a}$$

and for $x = 0$ and $x = l$

$$\Delta\varphi^{\bar{k}}_{z,\bar{p},k} = \frac{\Delta M_{z,\bar{k},k}\, l}{6EJ_{z,k}} C^{\bar{k}}_{z,\bar{p},k}$$

$$\Delta\varphi^{\bar{k}}_{z,\bar{k},k} = \frac{\Delta M_{z,\bar{k},k}\, l}{3EJ_{z,k}} C^{\bar{k}}_{z,\bar{k},k} \tag{27b}$$

where

$$C^{\bar{k}}_{z,\bar{p},k} = \frac{6EJ_{z,k}}{l^2}\sum_n \frac{\dfrac{n\pi\, \Delta N_k \sum_{m=1}^{k-1} f_{y,m,n}}{\Delta M_{z,\bar{k},k}} - 2\cos n\pi}{N_{kr,z,k,n} - N_k(1+\gamma_{y,k} N_{kr,z,k,n})}$$

$$C^{\bar{k}}_{z,\bar{k},k} = \frac{3EJ_{z,k}}{l^2}\sum_n \frac{\dfrac{n\pi\, \Delta N_k \sum_{m=1}^{k-1} f_{y,m,n}}{\Delta M_{z,\bar{k},k}} - 2\cos n\pi}{N_{kr,z,k,n} - N_k(1+\gamma_{y,k} N_{kr,z,k,n})}\cos n\pi \tag{27c}$$

In case of a initially deformed bar in compression, loaded with an uniformly distributed load of constant intensity as in Figure 6 the following is obtained

$$\Delta\varphi_{z,k}(x) = \sum_n \frac{n\pi}{l} \frac{\frac{4\Delta p_{y,k} l^2}{(n\pi)^3}\sin^2\frac{n\pi}{2} + M^N_{z,k,n}}{N_{kr,z,k,n} - N_k - \gamma_{y,k} N_k N_{kr,z,k,n}} \cos\frac{n\pi x}{l} \tag{28a}$$

where for $k = 1$ $M^N_{z,1,n} = N_1 f_{y,0,n}$ and for $k > 1$ $M^N_{z,k,n} = \Delta N_k \sum_{m=1}^{k-1} f_{y,m,n}$, hence, for $x = 0$ and for $x = l$

$$\begin{aligned} \Delta\varphi^q_{z,\bar{p},k} &= \sum_n \frac{n\pi}{l} \frac{\frac{4\Delta p_{y,k} l^2}{(n\pi)^3}\sin^2\frac{n\pi}{2} + M^N_{z,k,n}}{N_{kr,z,k,n} - N_k - \gamma_{y,k} N_k N_{kr,z,k,n}} \\ \Delta\varphi^q_{z,\bar{k},k} &= \sum_n \frac{n\pi}{l} \frac{\frac{4\Delta p_{y,k} l^2}{(n\pi)^3}\sin^2\frac{n\pi}{2} + M^N_{z,k,n}}{N_{kr,z,k,n} - N_k - \gamma_{y,k} N_k N_{kr,z,k,n}} \cos n\pi \end{aligned} \tag{28b}$$

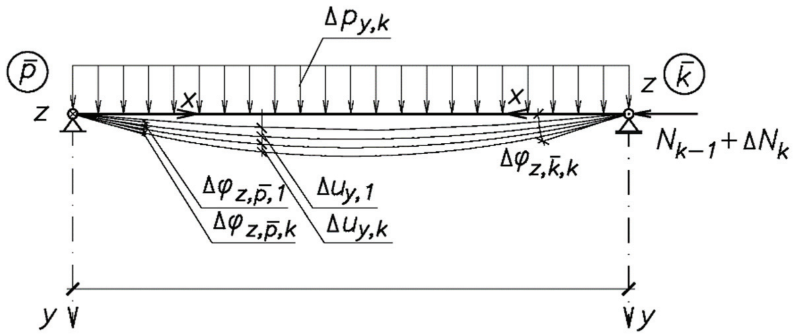

Fig. 6. A bar with hinged ends loaded uniformly by the load p_y

Similarly, it is possible to obtain transformation formulas of the displacement method for bending in the *xz* plane. It should be taken into account that with the coordinate system as in Figure 1, the positive value of the node moment $M_{y,\bar{p}}$ (with a vector compatible with the *y* axis) corresponds to the negative value of the elastic bend deflection and the positive value of the initial bend deflection.

Example

The solution presented earlier was supplemented with a module with rigidity degradation of thin-walled members exposed to normal stress, based on the EN 1993-1-3:2006 standard (European Committee for Standardization [CEN], 2006).This extended solution was used to analyse an individual arch of the umbrella roof covering with characteristics as in Figure 7. The supporting structure of the umbrella roof consists of columns fixed in foundations in both directions, spaced longitudinally every 6.0 m. Eave beams supported on two columns are designed as a box sections composed of two C-profiles 240. These beams were connected each other with tie rods with a diameter of 24 mm located in planes of the columns and in planes located 3 m away from them. The covering was made of arched steel sections with cross-sectional dimensions as in Figure 7, with a thickness of 1 mm and a width of 600 mm between axes of their locks. The umbrella roof was located in the third wind load zone and the fourth snow load zone according to the Polish National Annex to Eurocode 1 Part 4 (Polski Komitet Normalizacyjny [PKN], 2010).

The arch was treated as a system of pre-bent bar element in compression and bending, with compressive rigidity EA_k, bending rigidity EJ_k and shear rigidity $GA_{v,k}$ variable in subsequent incremental steps *k*. The

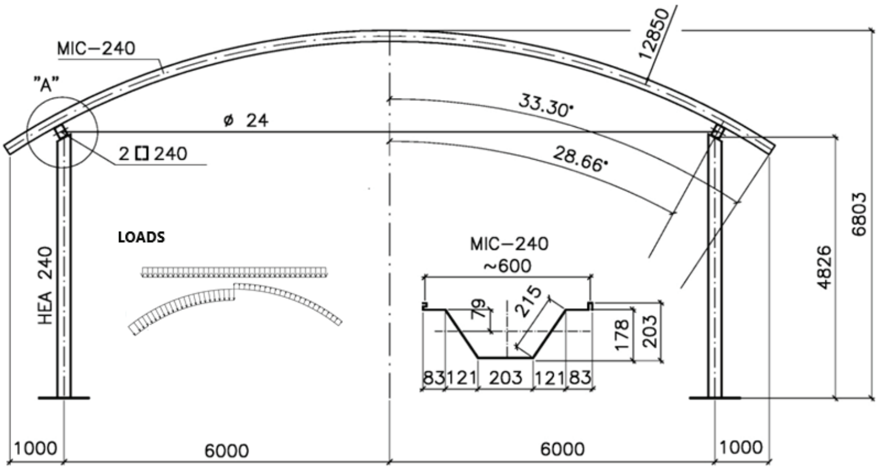

Fig. 7. Characteristics of the umbrella roof

value of the initial deflection was assumed consistent with deflection of the arch with a chord equal to the element length. In the solution, the effect of general instability is considered as $P - \Delta$ effect, i.e. the effect of changing the configuration of the bar members on the values of nodal shear forces. The impact of local instability of the walls was evaluated at each incremental step in a manner provided for in the PN-EN 1993--1-3:2008 standard (PKN, 2008).

Equations of displacement method derived as shown before were used to build a computer program for calculating any planar bar systems (Fig. 8). Due to

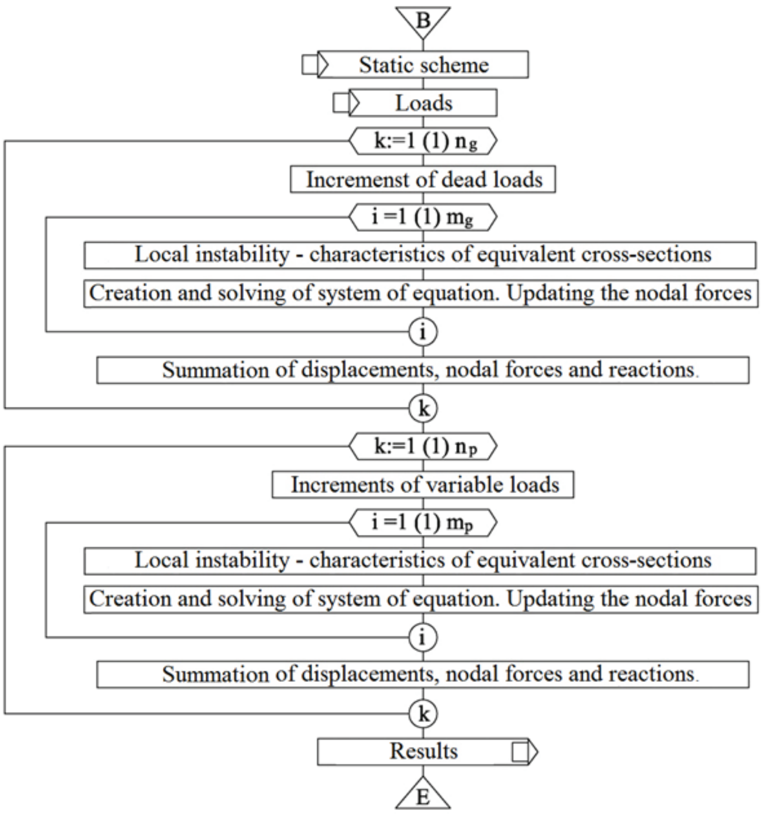

Fig. 8. Block diagram of the program for calculating arched trapezoidal corrugated sheets

capabilities of this software, two calculation schemes of the umbrella roof transverse system were considered. In the first scheme, considering eave beams non--deformable (i.e. rigid), 1/10 stiffness of columns and 1/5 stiffness of tie rods were assigned to the single arch sheet, which allowed to obtain reliable results for arches in the column area. The second scheme takes into account the additional impact of the eave beams deformability for corrugated arched sheets located in the area of the largest deflections of these beams.

The arch was divided into 14 elements. The results obtained for the intermediate arch (according to the second scheme) are presented in Figure 9 regarding to unloaded arch. The solid line shows the results obtained without the effect of local instability u_1, and the broken line takes into account these instability u_2. The sum of the x coordinates of the elements nodes and their displacements u_x was treated as the abscissae of the graph, and the u_y displacements of nodes as the ordinates.

Differences in the values of vertical displacements of arch nodes in the key exceed slightly 5% of displacements of the arc calculated without the effect of local instability of the walls.

CONCLUSIONS

The paper presents the incremental solution for a bar using the displacement method by the second order theory, which includes the possibility of changing the cross-section characteristics under the subsequent loads. This solution can be useful for both strengthened structures and structures consisting of thin--walled members, which walls lose their stability under increasing stress.

In engineering practice, the utilisation index of the strengthened members in bending and compression is calculated as the sum of the utilisation indexes of the member before and after strengthening. For example, in case symmetrical cross-section strengthening, in the first stage the $\Delta M_{z,1}(x) = \Delta M_{z,1}(p) + \Delta N_1 \cdot \Delta u_{y,1}$ moment is considered and in the second stage the $\Delta M_{z,2}(x) = \Delta M_{z,2}(p) + \Delta N_2 \cdot \Delta u_{y,2}$. Compared with Eq. (8a), the components $\Delta N_1 \cdot \Delta u_{y,2}$ and $\Delta N_2 \cdot \Delta u_{y,1}$ are thus omitted. In case of the load bearing capacity of members check according to following equation

$$\frac{\Delta N_{1,Ed}}{\chi_1 N_{1,Rc}} + \frac{\Delta M_{1,Ed}}{M_{1,Rd}} + \frac{\Delta N_{2,Ed}}{\chi_2 N_{2,Rc}} + \frac{\Delta M_{2,Ed}}{M_{2,Rd}} \leq 1 \quad (29)$$

the value of the bending moment calculated in the second stage on the basis of external loads should be increased by aforementioned omitted components. Then $\Delta M_{2,Ed} = \Delta M_2(p) + \Delta N_1 \cdot \Delta u_{y,2} + \Delta N \cdot \Delta u_{y,1}$.

Effectiveness of strengthening the actual structure may be checked, for example, by means of innovative strain and stress fiber Bragg grating (FBG) and residual magnetic field (RMF) analyses (Juraszek, 2019, 2020).

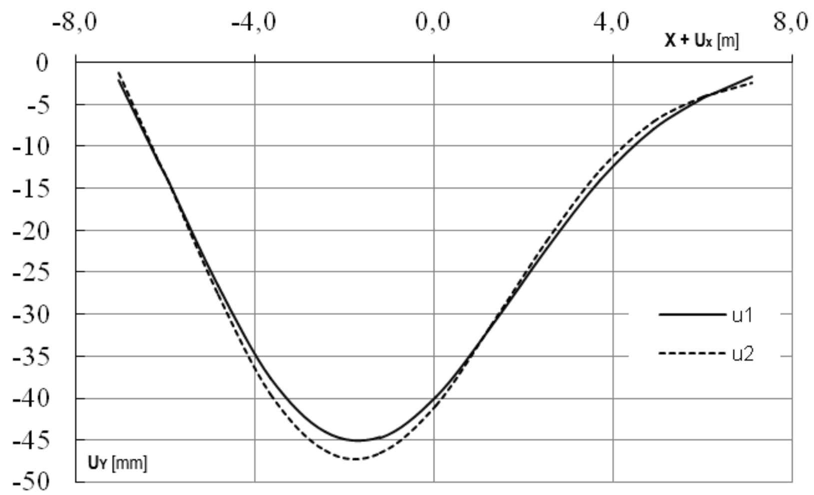

Fig. 9. Arc strains

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STATYKA ELEMENTU ZE ZMIENNYM PRZEKROJEM POD OBCIĄŻENIEM

STRESZCZENIE

W artykule przedstawiono rozwiązanie przyrostowe ściskanego i zginanego pręta metodą przemieszczeń z uwzględnieniem siły poprzecznej, w którym w poszczególnych krokach przyrostowych można zmieniać charakterystyki geometryczne przekroju łącznie ze zmianą położenia osi obojętnej. Takie rozwiązanie jest przydatne zarówno w obliczeniach statycznych konstrukcji z prętami wzmacnianymi, jak i konstrukcji, w których następuje degradacja sztywności pod obciążeniem na skutek utraty stateczności ścianek. W rozwiązaniu wykorzystano szeregi trygonometryczne do przedstawienia funkcji obciążeń, sił wewnętrznych i przemieszczeń. Do elementu mogą być przyłożone równomiernie rozłożone obciążenie oraz dowolna liczba sił skupionych i momentów przywęzłowych.

Słowa kluczowe: rozwiązanie przyrostowe metodą przemieszczeń, konstrukcje wzmacniane, degradacja sztywności elementów