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ORIGINAL PAPER

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ON THE METHOD OF CALCULATION OF BUCKLING AND POST-BUCKLING BEHAVIOR OF LAMINATED SHELLS WITH SMALL ARBITRARY IMPERFECTIONS

Mykola Semenyuk[1], Volodymyr Trach[2✉], Natalia Zhukova[1]

[1] S.P. Timoshenko Institute of Mechanics NAS Ukraine, Kyiv, Ukraine
[2] Institute of Civil Engineering, Warsaw University of Life Sciences – SGGW, Warsaw, Poland; National University of Water Management and Nature Resources Use, Rivne, Ukraine

ABSTRACT

In the present paper the variant of computational methods of stability and initial post-buckling behavior of isotropic shells was generalized with respect to laminated composite shells. Using methods of the asymptotic analysis of the Timoshenko–Mindlin theory, the relationships for calculation of shells with the small geometrical imperfections of the different shapes have been produced. On the basis of obtained equations, the technique of calculation of a non-linear pre-buckling state, limiting loads and bifurcation, and also initial post-buckling behavior of laminated cylindrical shells at an axial compression and external pressure have been worked out. The results of calculation for the shells made of glass fiber-reinforced plastic and carbon fiber-reinforced plastic with multimodal imperfections and dimple imperfections have been presented. The features of transformation of the interacting modes of imperfections have been researched.

Key words: buckling, post-buckling, imperfection, laminated shell, mode interaction, Cauchy problem

MODEL EQUATIONS

Let a shell be loaded by a system of forces proportional to a certain parameter λ. According to a Timoshenko–Mindlin theory, the generalized displacements of the shell are described by the vector $U = (u, v, w, \theta, \psi)$. Analogous vector notation for the generalized stresses and strains $\sigma = (T_{11}, T_{22}, S, T_{23}, T_{13}, M_{11}, M_{22}, M_{12}, M_{21})$, $\varepsilon = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}, \varepsilon_{23}, \varepsilon_{13}, k_{11}, k_{22}, \tau_1, \tau_2)$ is introduced here. If the field of initial imperfections is characterized by the vector $\tilde{u}$, then the following expressions for the deformations and generalized displacement can be derived

$$\varepsilon(u,\, \tilde{u}) = \varepsilon(u) + \varepsilon'' u\tilde{u};\;\; \Delta(u, \tilde{u}) = \Delta(u) + \Delta'' u\tilde{u} \quad (1)$$

The stress vector $\tilde{\sigma}$ for a shell with imperfections is related to the deformation vector $\varepsilon(u, \tilde{u})$ by the stiffness matrix for a laminated shell H

$$\tilde{\sigma} = H\varepsilon(u,\, \tilde{u}) \quad (2)$$

The principle of virtual work takes the form

$$\tilde{\sigma}\varepsilon'(u, \tilde{u})\delta u - \lambda\Delta'(u, \tilde{u})\delta u = 0 \quad (3)$$

Though modified, Eqs. (1)–(3) enable solving a variety of nonlinear problems on deformation of shells with initial geometric imperfections. The asymptotic method (Byskov & Hutchinson, 1997) to the Eqs. (1)–(3) has been applied here. It has been assumed

✉trach-vm@ukr.net

that the subcritical stress-deformed state of the shells is linear. For $\lambda = 1$, assume that the fields of displacements, deformations and stresses are characterized by the vectors u_0, ε_0, σ_0. After linearization of equations in the neighborhood of the bifurcation load in aim to determine the critical values of λ_i and the corresponding buckling modes u_i, one obtains a set of equations

$$\sigma_i \varepsilon'(0)\delta u + \lambda_i \sigma_0 \varepsilon'' u_i \delta u - \lambda_i \Delta'' u_i \delta u = 0 \quad (4)$$

$$\sigma_i = H\varepsilon_i \quad (5)$$

$$\varepsilon_i = \varepsilon'(0)u_i; \quad i = 1, \ ..., \ M \quad (6)$$

These M forms are orthogonal

$$\sigma_0 \varepsilon'' u_i u_j - \Delta'' u_i u_j = 0, \qquad i \neq j \quad (7)$$

The amplitudes ξ_i of the modes u_i remain indeterminate during solution of the homogeneous problem (4)–(6) and can be found only by solving the original non-linear problem (1)–(7). The displacement vector can be represented as the asymptotic expansion

$$u = \lambda u_0 + \xi_i u_i + \xi_i \xi_j u_{ij} \quad (8)$$

Here and further the rule of summing over repeated indices is adopted. The smallness of parameters ξ_i follows from the fact that $\xi_i \to 0$ as $\lambda \to \lambda_i$. The displacements u_{ij} are orthogonal to the buckling modes

$$\sigma_0 \varepsilon'' u_i u_{kl} - \Delta'' u_i u_{kl} = 0;$$
$$i = 1, ..., M; \ k = 1, ..., M; \ l = 1, ..., M \quad (9)$$

Substituting (8) into (4), one obtains

$$\lambda\sigma_0 \varepsilon'(0)\delta u - \lambda\Delta'(0)\delta u +$$
$$+\xi_i \left[\sigma_i \varepsilon'(0)\delta u + \lambda\sigma_0 \varepsilon'' u_i \delta u - \lambda\Delta'' u_i \delta u\right] +$$
$$+\xi_i \xi_j \left[\sigma_{ij} \varepsilon(0)\delta u + \sigma_i \varepsilon'' u_j \delta u + \lambda\sigma_0 \varepsilon'' u_{ij} \delta u - \lambda\Delta'' u_{ij} \delta u\right] +$$
$$+\xi_i \xi_j \xi_k \left[\sigma_i \varepsilon'' u_{jk} \delta u + \sigma_{ij} \varepsilon'' u_k \delta u + H\varepsilon'' u_i u_{jk} \varepsilon'(0)\delta u\right] + ... = 0$$
$$(10)$$

Here, terms of higher order in ξ have been omitted and, moreover

$$\sigma_{ij} = H\varepsilon_{ij}, \qquad \varepsilon_{ij} = \varepsilon'(0)u_{ij} + \frac{1}{2}\varepsilon'' u_i u_j \quad (11)$$

Because $\lambda\sigma_0\varepsilon'(0)\delta u - \lambda\Delta'(0)\delta u = 0$, basing on (4), the expression (10) can be reduced to the form

$$\xi_i \left(\lambda - \lambda_i\right)\left(\sigma_0 \varepsilon'' u_i \delta u - \Delta'' u_i \delta u\right) +$$
$$+\xi_i \xi_j L_{ij}(\lambda) + \xi_i \xi_j \xi_k L_{ijk} + ... = 0 \quad (12)$$

where:

$$L_{ij}(\lambda) = \sigma_{ij} \varepsilon'(0)\delta u + \sigma_i \varepsilon'' u_j \delta u + \lambda\sigma_0 \varepsilon'' u_{ij} \delta u - \lambda\Delta'' u_{ij} \delta u$$
$$L_{ijk} = \sigma_i \varepsilon'' u_{jk} \delta u + \sigma_{ij} \varepsilon'' u_k \delta u + H\varepsilon'' u_i u_{jk} \varepsilon'(0)\delta u$$
$$(13)$$

A variational equation in the fields u_{ij}, ε_{ij}, σ_{ij} can be obtained from Eq. (12), when it has been taken into account that the variation δu in this case satisfies the orthogonality condition

$$\sigma_0 \varepsilon'' u_i \delta u - \Delta'' u_i \delta u = 0 \quad (14)$$

Neglecting, in Eq. (12), terms of third or higher order in ξ_i, the symmetry of the desired functions relative to the indices i, j results in an equation

$$\sigma_{ij} \varepsilon'(0)\delta u + \lambda\sigma_0 \varepsilon'' u_{ij} \delta u - \lambda\Delta'' u_{ij} \delta u =$$
$$= -\frac{1}{2}\left(\sigma_i \varepsilon'' u_j \delta u + \sigma_j \varepsilon'' u_i \delta u\right) \quad (15)$$

In contrast to Koiter's classical approach (Koiter, 1963), which is based on the Lyapunov–Schmidt method, the method proposed by Byskov and Hutchinson (1977) does not use expansion of the coefficients of $L_{ij}(\lambda)$ in Tailor series in the parameter $(\lambda - \lambda_j)$. As a result, the solution of Eq. (15) depends on the load λ. Koiter (1963) proposed a use of the smallest λ_j for the λ in Eq. (15). However, the problems under consideration make it necessary to determine this inaccuracy. If the homogeneous boundary-value problem (4), (5) and inhomogeneous problem (11), (15) are resolved, then, for the amplitudes ξ_i from equation (12) (where $\delta u = u_1$ is set), one can derive the following system of non-linear equations:

$$\xi_r \left(1 - \frac{\lambda}{\lambda_r}\right) + \xi_i \xi_j a_{ijr} + \xi_i \xi_j \xi_k b_{ijkr} = \bar{\xi}_r \frac{\lambda}{\lambda_r}; \quad r = 1, \ ..., M \quad (16)$$

where:

$$a_{ijr} = -\frac{A_{ijr}}{2D}; \ b_{ijr} = -\frac{B_{ijr}}{D}$$

$$A_{ijr}=\sigma_r\varepsilon''u_iu_j+2\sigma_i\varepsilon''u_ju_r,\ D=\lambda_r\left(\sigma_0\varepsilon''{u_r}^2-\Delta''{u_r}^2\right)$$

$$B_{ijkr}=\sigma_i\varepsilon''u_ru_{jk}+\sigma_{ij}\varepsilon''u_ku_i+\sigma_r\varepsilon''u_iu_{jk} \tag{17}$$

Equation (16) can be used to investigate nonlinear deformations of imperfect structures in the subcritical state, for computation of critical (bifurcation or limiting) loads and for computation of the initial post-buckling behavior of the shells under discussion.

In general, the representation (16) is uniformly valid whether the modes are simultaneous, nearly simultaneous, or well separated.

This property of the Byskov–Hutchinson method is used below for development of a technique of calculation of stability and post-buckling behavior of laminated composite cylindrical shells with multimodal imperfections, in particular, by imperfections which are approximated by trigonometric Fourier series.

THE SOLUTION FOR THE CYLINDRICAL SHELLS

The relationships obtained have been used for buckling analysis of a multilayer cylindrical composite shells with geometrical imperfections. The imperfections' shape is described by trigonometric polynomials (fragmenting of Fourier series). According to the basic statements of the Byskov–Hutchinson method (1997) and Byskov method (2004) relationships will be presented which are necessary for solving with use of the Timoshenko–Mindlin non-linear shell theory (Vanin & Semenyuk, 1987). The expression of virtual work principle in this case is

$$\int_0^L\int_0^{2\pi R}\Big[T_{11}\delta\varepsilon_1+T_{22}\delta\varepsilon_2+T_{12}^*\delta\omega_1+T_{21}^*\delta\omega_2+ \\ +T_{13}^*\delta\theta_1+T_{23}\delta\theta_2+T_{13}\delta\theta+T_{23}^*\delta\psi+ \\ +M_{11}\delta k_1+M_{22}\delta k_2+M_{12}\delta t_1+ \\ +H\left(\delta t_1+\delta t_2\right)dxdy-\delta A\Big]=0 \tag{18}$$

where:

L, R – length and radius of a shell,

x, y – coordinate lines which coincide with a generatrix and directrix of a cylindrical reduced surface,

T_{ij}, M_{ij} – efforts and moments, which are statically equivalent to actual stresses.

$$T_{12}^*=S+T_{11}\omega_1;\ T_{21}^*=S+\frac{1}{R}H+T_{22}\omega_2$$

$$T_{13}^*=T_{13}+T_{11}\theta_1+S\theta_2;\ T_{23}^*=T_{23}+S\theta_1+T_{22}$$

The strains are given by

$$\varepsilon_{11}=\varepsilon_1+\frac{1}{2}\left({\omega_1}^2+{\theta_1}^2\right);\quad \varepsilon_{22}=\varepsilon_2+\frac{1}{2}\left({\omega_2}^2+{\theta_2}^2\right)$$

$$\varepsilon_{12}=\omega_1+\omega_2+\theta_1\theta_2$$

$$k_{11}=k_1;\quad k_{22}=k_2+\frac{\varepsilon_2}{R};\quad k_{12}=t_1+t_2+\frac{\omega_2}{R}$$

$$\varepsilon_{13}=\theta+\theta_1;\quad \varepsilon_{23}=\psi+\theta_2$$

where:

$$\varepsilon_1=\frac{\partial u}{\partial x};\quad \omega_1=\frac{\partial v}{\partial x};\quad \theta_1=\frac{\partial w}{\partial x}$$

$$\varepsilon_2=\frac{\partial v}{\partial y}-\frac{w}{R};\quad \omega_2=\frac{\partial u}{\partial y};\quad \theta_2=\frac{\partial w}{\partial y}+\frac{v}{R}$$

$$k_1=\frac{\partial\theta}{\partial x};\quad k_{22}=\frac{\partial\psi}{\partial y};\quad t_1=\frac{\partial\psi}{\partial x};\quad t_2=\frac{\partial\theta}{\partial y}$$

The variation of work of external loads λA has a different kind depending on character of their distribution on a frontal or on a lateral surface.

In case of a cylindrical shell, one of the most practically interesting is the version of loading by longitudinal compressive forces – T_{11}^0

$$\delta A=-\int_0^{2\pi R}T_{11}^0\delta u\Big|_0^L dy$$

or uniform external pressure with intensity q

$$\delta A=\int_0^L\int_0^{2\pi R}q\left[-\theta_1\delta u-\theta_2\delta v+\left(1+\varepsilon_1+\varepsilon_2\right)\delta w\right]dxdy$$

The initial imperfections are characterized by a deflection $w_0(x, y)$.

It is supposed, that the shell consists of N layers of a fibrous composite material which have a symmetrical structure with regard to a middle surface. The relationships of elasticity in this case may be written as

$$T_{11} = C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22}\,;\;\; T_{22} = C_{12}\varepsilon_{11} + C_{22}\varepsilon_{22}\,;\;\; S = C_{66}\varepsilon_{12}\,;\;\; T_{13} = C_{55}\varepsilon_{13}\,;\;\; T_{23} = C_{44}\varepsilon_{23}$$

$$M_{11} = D_{11}k_{11} + D_{12}k_{22}\,;\;\; M_{22} = D_{12}k_{11} + D_{22}k_{22}\,;\;\; H = D_{66}k_{12}$$

in which the rigidities C_{ij} and D_{ij} are given by

$$C_{kl} = \sum_{i=1}^{N} C_{kl}^{i}\,;\;\; D_{kl} = \sum_{i=1}^{N}\left(D_{kl}^{i} + z_i^2 C_{kl}^{i}\right)$$

where

z_i is coordinate of a middle surface of i-th layer.

The linearized equations for a cylindrical shell may be expressed as

$$\int_0^L \int_0^{2\pi R} \Big\{ T_{11}^{i}\delta\varepsilon_1 + S^{i}\delta\omega_1 + T_{22}^{i}\delta\varepsilon_2 + \left(S^{i} + \frac{2}{R}H^{i}\right)\delta\omega_2 + T_{13}^{i}\left(\delta\theta_1 + \delta\theta\right) +$$
$$+ H^{i}\delta k_{12} + T_{23}^{i}\left(\delta\theta_2 + \delta\psi\right) + M_{11}^{i}\delta k_{11} + M_{22}^{i}\delta k_{22} + \lambda_i\left[T_{11}^{0}\left(\omega_1^{i}\delta\omega_1 + \theta_1^{i}\delta\theta_1\right) + \right. \tag{19}$$
$$\left. + T_2^{0}\left(\omega_2^{i}\delta\omega_2 + \theta_2^{i}\delta\theta_2\right) + S^{0}\left(\theta_1^{i}\delta\theta_2 + \theta_2^{i}\delta\theta_1\right)\right] \Big\}\,dxdy = 0$$

Variational equation (19) relative to variables of second order in the Timoshenko–Mindlin shell theory is represented as

$$\int_0^L \int_0^{2\pi R} \Big\{ T_{11}^{ij}\delta\varepsilon_1 + S^{ij}\delta\omega_1 + T_{22}^{ij}\delta\varepsilon_2 + \left(S^{ij} + \frac{2}{R}H^{ij}\right)\delta\omega_2 + T_{13}^{ij}\left(\delta\theta_1 + \delta\theta\right) + T_{23}^{ij}\left(\delta\theta_2 + \delta\psi\right) + M_{11}^{ij}\delta k_{11} + M_{22}^{ij}\delta k_{22} +$$
$$+ H^{ij}\delta k_{12} + \lambda_i\left[T_{11}^{0}\left(\omega_1^{ij}\delta\omega_1 + \theta_1^{ij}\delta\theta_1\right) + T_2^{0}\left(\omega_2^{ij}\delta\omega_2 + \theta_2^{ij}\delta\theta_2\right) + S^{0}\left(\theta_1^{ij}\delta\theta_2 + \theta_2^{ij}\delta\theta_1\right)\right] \Big\}\,dxdy =$$
$$= -\frac{1}{2}\int_o^L \int_0^{2\pi R} \left[\left(T_{11}^{i}\omega_1^{ji} + T_{11}^{j}\omega_1^{i}\right)\delta\omega_1 - \frac{1}{2}\left(T_{22}^{i}\omega_2^{j} + T_{22}^{j}\omega_2^{i}\right)\delta\omega_2 - \frac{1}{2}\left(T_{11}^{i}\theta_1^{j} + S^{i}\theta_2^{j} + T_{11}^{j}\theta_1^{i} + S^{j}\theta_2^{i}\right)\delta\theta_1 - \right. \tag{20}$$
$$\left. -\frac{1}{2}\left(T_{22}^{i}\theta_2^{j} + S^{i}\theta_2^{j} + T_{22}^{j}\theta_{21}^{i} + S^{j}\theta_2^{i}\right)\delta\theta_2\right]dxdy$$

If the solutions of boundary value problems (4) and (15) are obtained, then the coefficients a_{ijr} and b_{ijkr} of system of Eq. (16) are determined by Eq. (17).

The stated technique has been applied for the calculation of laminated cylindrical shells having a total thickness t at axial compression and under external pressure. The displacements are adopted as the resolvents. In a solution of the homogeneous problem (4), the displacements are represented by trigonometric series, each term of which separately satisfies simple support conditions at the ends:

$$u_i = A^i_{m,n} \cos l_m \xi \cos n_i \phi$$
$$v_i = B^i_{m,n} \sin l_m \xi \sin n_i \phi$$
$$w_i = C^i_{m,n} \sin l_m \xi \cos n_i \phi \quad (21)$$
$$\theta_i = D^i_{m,n} \cos l_m \xi \cos n_i \phi$$
$$\psi_i = E^i_{m,n} \sin l_m \xi \sin n_i \phi$$

where:

$$l_m = \frac{m\pi R}{L}; \quad \xi = \frac{x}{R}; \quad \varphi = \frac{y}{R}.$$

The system of homogeneous algebraic equations is obtained by substituting (21) into (19). An exhaustive sequential inspection of the mode-generation parameters m and n yields the spectrum of eigenvalues λ_i and corresponding eigenvectors, which are normalized in such a way as to make $C^i_{m,n} i = 1$.

The solution of the system of Eq. (12) with allowance for the form of their right-hand sides and simple supports at the ends is written

$$u_{ij} = \sum_k \left[A^{ij}_{k,1} \cos(n_i - n_j)\varphi + A^{ij}_{k,2} \cos(n_i + n_j)\varphi\right] \cos l_k \xi$$

$$v_{ij} = \sum_k \left[B^{ij}_{k,1} \sin(n_i - n_j)\varphi + B^{ij}_{k,2} \sin(n_i + n_j)\varphi\right] \sin l_k \xi$$

$$w_{ij} = \sum_k \left[C^{ij}_{k,1} \cos(n_i - n_j)\varphi + C^{ij}_{k,2} \cos(n_i + n_j)\varphi\right] \sin l_k \xi$$

$$\theta_{ij} = \sum_k \left[D^{ij}_{k,1} \cos(n_i - n_j)\varphi + D^{ij}_{k,2} \cos(n_i + n_j)\varphi\right] \cos l_k \xi$$

$$\psi_{ij} = \sum_k \left[E^{ij}_{k,1} \sin(n_i - n_j)\varphi + E^{ij}_{k,2} \sin(n_i + n_j)\varphi\right] \sin l_k \xi$$

(22)

The system of Eq. (16) is solved for a given value of the load parameter λ Newton's method. A stepped procedure has been considered. The solution obtained in the i-th loading step is used as the initial approximation for the next (i + 1)-st step. The following procedure (Bazhenov, Semenyuk & Trach 2010) is used to construct the solution at points where the Jacobian of the system (16) is equal to zero. We introduce an (M + 1)-dimensional vector $\overline{X}$ with components $(\xi_1, ..., \xi_M, \lambda)^T$. The system (16) can be written in the compact form

$$F_r\left(\overline{X}\right) = 0; \quad r = 1, \ ..., M \quad (23)$$

Differentiating Eq. (21) with respect to the parameter ξ, which corresponds to progression along the curve of solutions of the system, a system of M linear ordinary differential equations at (M + 1) unknowns can be obtained

$$\sum_{j=1}^{M+1} F_{r,j} \frac{d\xi_j}{ds} = 0; \quad r = 1, \ ..., M \quad (24)$$

where $\overline{J} = \left[F_{r,j}\right] = \left[\frac{\partial F_r}{\partial \xi_i}\right]$ is the Jacobian matrix of the system (23). The Jacobian has the rank $\left[\overline{J}\right] = M$ at regular and limit points. The solution of the system (24) can be written in the form of a Cauchy problem:

$$\frac{d\overline{X}}{ds} = ort\left(J, Q\right)$$

basing on the initial condition $\overline{X}\left(s_0\right) = \overline{X}_0$.

The operator $ort(J, Q)$ denotes the process of orthogonalization of row vectors of the matrix $\overline{J}$ and determination of the unit vector that completes the original basis to an (M + 1)-dimensional basis (Bazhenov et al., 2010). The initial value $\overline{X}\left(s_0\right)$ is found by solving the system (23) according to Newton's method for $\lambda << \lambda_s$ (limiting value).

NUMERICAL ANALYSIS

With use of the above-mentioned calculation technique, some features of the nonlinear deformation of composite cylindrical shells with geometric imperfection in the form of superposition of the eigenmodes of the initial linearized problem have been investigated. Consider a shell of medium thickness (R/t = 30/1.6), medium length (L/R = 2), consisting of 16 individual layers, which are reinforced fiberglass or carbon fiber. In the examples below, the distribution of reinforcement directions through the thickness is supposed as (0°, 45°, –45°, 90°, …). The layer is assumed to have the following mechanical characteristics:

- for carbon fiber: E_{11} = 0.161·10^6 MPa, E_{22} = = 0.115·10^5 MPa, G_{12} = 0.717·10^4 MPa, G_{13} = = 0.717·10^4 MPa, G_{23} = 0.7·10^4 MPa, ν_{12} = 0.349;
- for fiberglass: E_{11} = 0.454·10^5 MPa, E_{22} = 0.109·10^5 MPa, G_{12} = 0.424·10^4 MPa, G_{13} = 0.436·10^4 MPa, G_{23} = 0.384·10^4 MPa, ν_{12} = 0.26.

For estimation of reliability of the results obtained has been assumed in Eq. (16) that r = 1 and the equilibrium curves have been compared with those that correspond to the equations by Koiter (1963):

$$\xi\left(1-\frac{\lambda}{\lambda_c}\right)+\xi^3 b=\bar{\xi}\,\frac{\lambda}{\lambda_c} \qquad (25)$$

Shown in Figure 1, Curves 1 and 2 are obtained by Eqs. (16) and (25) in the case of an axial compression of fiberglass shells. The closeness of these curves indicates the possibility of calculation of post-buckling behavior of the considered shells with use of Eq. (16) or (25), according to the shape of imperfections.

Taking into account the interaction of several modes, it has been examined the kind of equilibrium curve $\xi\left(\frac{\lambda}{\lambda_c}\right)$ at q = 1, depending on the number of modes. The numbering of the curves in Figure 2 corresponds to the number of modes.

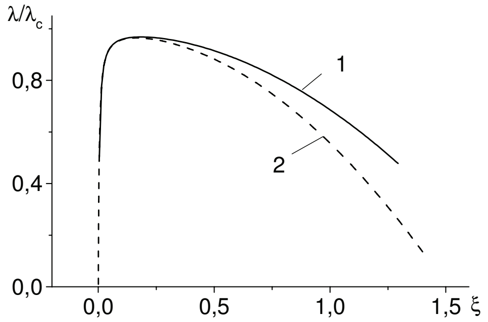

Fig. 1. The curves obtained by Eqs. (16) and (25) in the case of an axial compression of fiberglass shells

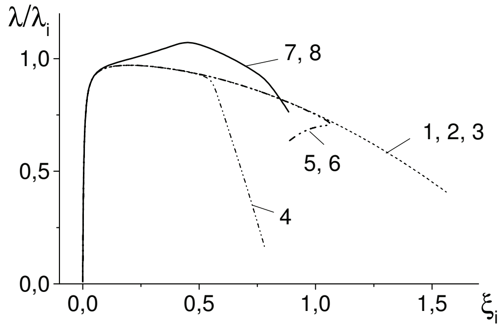

Fig. 2. The kind of equilibrium curve depending on the number of modes

As it is seen, the additions of imperfections of the second and the third modes do not change Curve 1 obtained only at presence of the first mode. The indicated modes are passive with respect to the first mode. The shape of Curve 4 shows that fourth mode is active. Taking this mode into account essentially changes the form of the equilibrium curve. At consequent increase of the number of modes, their considerable activity also appears (Curves 5–7).

Figure 3 shows the curves which illustrate one more feature of deformation of shells with multimodal imperfections Number of curves in this figure correspond to the indexes of unknown amplitudes ξ_i in the system of Eq. (16).

In spite of the fact that the initial values of all amplitudes ξ_i n a considered example are identical, their dependence on the load parameter λ has the brightly expressed disproportionate character. They become essentially different with increasing number *i*.

Representation of solutions as a Fourier series in terms of the eigenmodes of the linearized problem has been used to study the non-linear deformation of composite shells with local geometrical imperfections. It has been supposed that the shell surface has the dimple shaped initial imperfections given by (Amazigo & Fraser, 1971)

$$w_0\left(\xi,\varphi\right)=\varepsilon\, e^{-\left|\frac{\varphi}{\varphi_1}\right|}\sin l_m\xi \qquad (26)$$

where ε is a small parameter, $-\varphi_1 \le \varphi \le \varphi_1$; $m = 1, \ldots$

The function $e^{-\left|\frac{\varphi}{\varphi_1}\right|}$ is even. Its expansion in a Fourier series can be expressed as

$$\frac{a_0}{2}+\sum_n a_n \cos n\varphi \qquad (27)$$

where:

$$a_0=\frac{2}{\pi}\varphi_1\left(1-e^{-1}\right);$$

$$a_n=\frac{2}{\pi}\frac{\varphi_1}{1+n^2\varphi_1^2}\left[1+e^{-1}\left(n\varphi_1\sin n\varphi_1-\cos n\varphi_1\right)\right].$$

Basing on the expansion (26), the initial deflection can be written as

$$w_0\left(\xi,\varphi\right)=\bar{\xi}_i w_i$$

where $\bar{\xi}_i=\varepsilon\, a_i$; w_i is *i*-th eigenmode, corresponding to the eigenvalue λ_i. If the initial values of amplitudes $\bar{\xi}_i$ and the critical parameters λ_i are known, then there is a possibility to explore the process of deformation of the shell, which has a small dimple depth.

It is necessary to note that the further computing result are obtained basing on the 19 terms of a Fourier series in the expression (27).

Figure 4 shows the equilibrium curves for a fiberglass shell of the assumed sizes at $\varphi_1 = \pi/9$ under external pressure. Curve 1 is obtained at $\varepsilon = 0.05$, Curve 2 at $\varepsilon = 0.1$, Curve 3 at $\varepsilon = 0.5$.

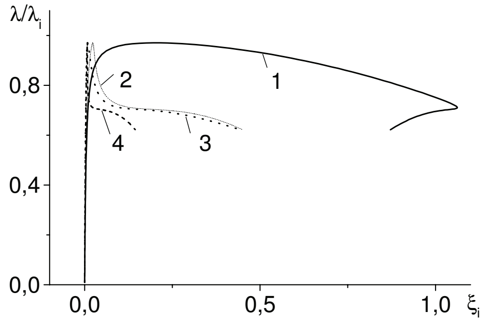

Fig. 3. The curves illustrating one more feature of deformation of shells with multimodal imperfections

Figures 5 and 6 show the dependence $\xi\left(\frac{\lambda}{\lambda_i}\right)$ for the fiberglass (Fig. 5) and carbon fiber (Fig. 6) shells with the same dimples at axial compression. These curves essentially differ from those shown in Figure 4, just as it takes place in the non-linear range of the theory of shells (Koiter, 1976).

It has been shown that the combination of an amplitude modulation by Koiter and asymptotic procedure by Byskov–Hutchinson is an effective mean of the solution of some non-linear problems.

CONCLUSIONS

Using methods of the asymptotic analysis of the Timoshenko–Mindlin theory, the relationships for calculation of shells with the small geometrical imperfections of the different shapes have been produced. On

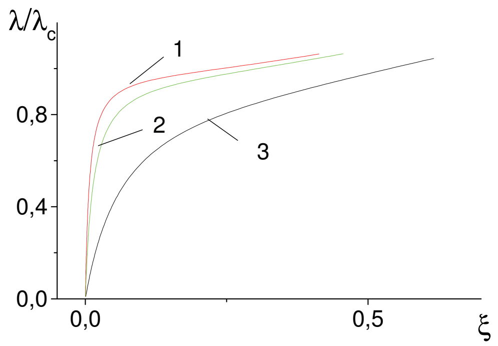

Fig. 4. The equilibrium curves for a fiberglass shell of the assumed sizes at under external pressure

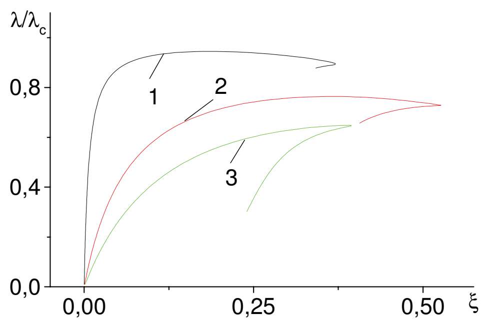

Fig. 5. The dependence for the fiberglass shells

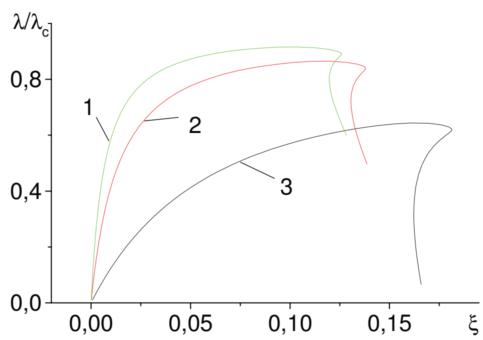

Fig. 6. The dependence for the carbon fiber shells

the basis of the obtained equations, the technique of calculation of a non-linear pre-buckling state, limiting loads and bifurcation, and also initial post-buckling behavior of laminated cylindrical shells at an axial compression and external pressure has been worked out. The results of calculation for the shells made of glass fiber-reinforced plastic and carbon fiber-reinforced plastic with multimodal imperfections and dimple imperfections have been presented.

Authors' contributions

Conceptualization: M.S. and V.T.; methodology: V.T.; validation: M.S. and V.T.; formal analysis: M.S. and V.T.; investigation: V.T.; resources: M.S.; data curation: N.Z.; writing – original draft preparation: N.Z.; writing – review and editing: V.T. and M.S.; visualization: N.Z.; supervision: V.T. and M.S.; project administration: M.S.; funding acquisition: M.S. and V.T.

All authors have read and agreed to the published version of the manuscript.

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O METODZIE OBLICZANIA WYBOCZENIA I STANU POWYBOCZENIOWEGO DLA POWŁOK Z MAŁYMI IMPERFEKCJAMI

STRESZCZENIE

W niniejszym artykule uogólniono wariant metody obliczania stateczności i początkowego stanu powyboczeniowego powłok izotropowych w odniesieniu do powłok kompozytowych. Z wykorzystaniem metody analizy asymptotycznej teorii Timoszenki–Mindlina stworzono związki do obliczeń powłok z małymi imperfekcjami geometrycznymi o różnych kształtach. Na podstawie otrzymanych równań opracowano technikę obliczeń nieliniowego stanu przed wyboczeniem, obciążeń granicznych i bifurkacji oraz początkowego stanu powyboczeniowego warstwowych powłok cylindrycznych przy ściskaniu osiowym i ciśnieniu zewnętrznym. Przedstawiono wyniki obliczeń powłok wykonanych z tworzywa sztucznego wzmacnianego włóknem szklanym i włóknem węglowym z imperfekcjami wielopostaciowymi i wgłębieniami. Zbadano cechy przekształcenia współdziałających postaci wyboczenia.

Słowa kluczowe: wyboczenie, stan powyboczeniowy, imperfekcje, powłoki warstwowe, współdziałanie postaci wyboczenia, problem Cauchy'ego