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A TWO-STEP ASYMPTOTIC MODELLING OF THE HEAT CONDUCTION IN A FUNCTIONALLY GRADED STRATIFIED LAYER

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Abstract. The aim of the contribution is to obtain heat conduction equations with slowly varying coefficients for a functionally graded stratified layer. The proposed procedure is realized in two steps. In the first step we deal with two independent isotropic materials, each of them reinforced by a periodically spaced very thin layers. Using the homogenization procedure we obtain two new orthotropic materials. In the second step these new materials are combined together in order to derive the functionally graded composite model constituting the basis for analysis.

Key words: material FGM, transversal gradation of effective properties, heat conduction, homogenization

INTRODUCTION

The general idea of the proposed approach is slightly related to what is called the reiterated homogenization as well as to the non uniform homogenization which are applied in the well known asymptotic procedures [Bensoussan et al. 1978, Lewiński and Telega 2000].

In contrast to periodic material structures, which were analyzed in the above mentioned contributions, this paper concerns functionally graded layered composite (FGM – Functionally Graded Materials). The general idea of these composites was discussed in the review paper [Suresh and Mortensen 1998] and some specific cases of FGM were studied in monographs [Woźniak C. et al. 2008, 2010].

The starting point of the analysis are two microlayered composite materials, each of them made of a certain basic material and reinforcement material both of them being periodically spaced. Fragments of the above mentioned composites are shown in Figure 1. It

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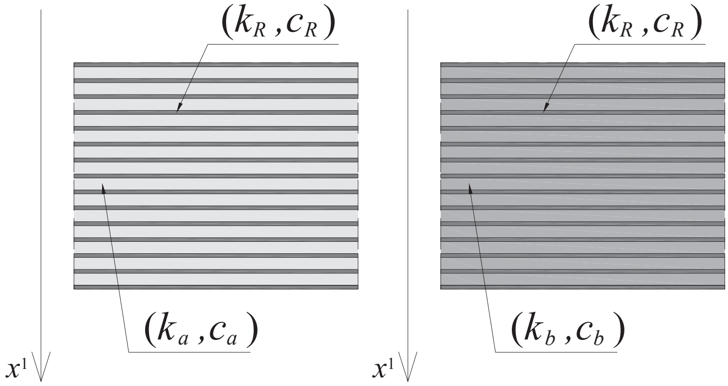

Fig. 1. Scheme of material "*a*" and material "*b*" reinforced by a system of periodically spaced inclusions made of material "*R*"

Rys. 1. Schemat materiału „*a*" i materiału „*b*" wzmocnionych periodycznie rozmieszczonymi wtrąceniami z materiału „*R*"

is assumed that materials "*a*", "*b*", "*R*" are isotropic heat conductors with heat conduction coefficients k_a, k_b, k_R, and specific heats c_a, c_b, c_R, respectively.

The fractions of reinforcements in the above composites are the same and will be denoted by ν, the fraction of material "*a*" and "*b*" are also the same and equal to 1 – ν, $\nu \in (0, 1)$. Introducing the orthogonal Cartesian coordinates $Ox^1x^2x^3$ we shall assume that the above two new materials are oriented in physical space along Ox^1 axis as normal to interfaces between basic materials ("*a*" or "*b*") and reinforcement material "*R*". Due to the well known homogenization procedure we deal now with two new composite materials. These materials will be denoted by "*A*" and "*B*" and theirs heat conduction coefficients along Ox^1 axis are given by:

$$K_A^1 = \frac{k_a k_R}{k_a \nu + k_R(1-\nu)} \tag{1}$$

$$K_B^1 = \frac{k_b k_R}{k_b \nu + k_R(1-\nu)}$$

At the same time the heat conduction coefficients in the direction of arbitrary axis which are normal to Ox^1axis are equal to:

$$\begin{aligned} K_A^2 &= K_A^3 = K_A = (1-\nu)k_a + \nu k_R \\ K_B^2 &= K_B^3 = K_B = (1-\nu)k_b + \nu k_R \end{aligned} \tag{2}$$

At the same time the specific heats of the composite "*A*", "*B*" materials under consideration are:

$$
\begin{aligned}
C_A &= (1-\nu)c_a + \nu c_R \\
C_B &= (1-\nu)c_b + \nu c_R
\end{aligned} \tag{3}
$$

It has to be emphasized that in the subsequent analysis we shall deal with two new materials which were denoted by “*A*”, “*B*”. Obviously these materials are not isotropic but have material characteristics in the heat conduction problems given by formulas (1), (2), (3).

FORMATION OF FGM LAYER

The object of our consideration is a layer occupying region $\Omega = (0,L)\times\Xi$, where Ξ is a region on Ox^2x^3 plane.

Let p be a natural number which is sufficiently large, i.e. $\frac{1}{p} << 1$. Let us assume that the layer under consideration is divided into p sublayers of the same thickness l, where $l = \frac{L}{p}$.

Every sublayer occupies the region $((n-1)l, nl)\times\Xi$, where n = 1, ... , p. Let $x^1 = x_n = \left(\frac{l}{2} + (n-1)l\right)\times\Xi$ stand for a midplane of the n-th layer. To ever n-th sublayer n = 1, ..., p there are uniquely assigned two fractions $\varphi_A\left(\frac{l}{2} + (n-1)l\right)$, $\varphi_B\left(\frac{l}{2} + (n-1)l\right)$, $\varphi_A(\cdot) + \varphi_B(\cdot) = 1$ such that the sublayer $\left((n-1)l, (n-1)l + \varphi_A\left(\frac{l}{2} + (n-1)l\right)l\right)\times\Xi$ is occupied by the *A*-th material and sublayer $\left((n-1)l + \varphi_A\left(\frac{l}{2} + (n-1)l\right)l, nl\right)\times\Xi =$ $= \left(nl - \varphi_B\left(\frac{l}{2} + (n-1)l\right)l, nl\right)\times\Xi$ is occupied by the *B*-th material.

The first important statement leading to the concept of functionally graded stratified material is there exist differentiable functions $\varphi_A(\cdot), \varphi_B(\cdot) \in C^1([0,L])$ which are slowly varying, $\varphi_A(\cdot) + \varphi_B(\cdot) = 1$.

Obviously these functions have a physical sense only for x^1 belonging to the midplanes of layers. The example of this structure of the layer is shown in Figure 2.

The second important statement concerns the compatibility between homogenized materials “*A*”, “*B*” and the reinforcement material “*R*”. It means that if the period of unhomogeneity of material “*a*” (“*b*”) with reinforcement material “*R*” is ξ then $\xi << \min_{n=1,\ldots,p} \varphi_A(x_n)l$. Generally speaking we deal here with two kinds of heterogeneity i.e. heterogeneity of composite materials *A*, *B*, and functionally graded materials under consideration.

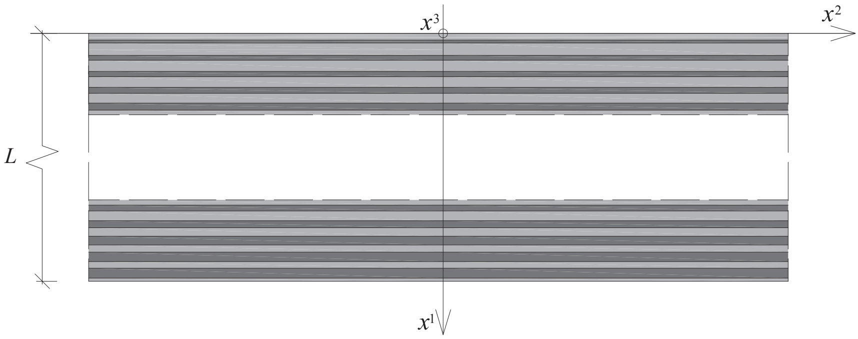

Fig. 2. Scheme of structure of the layer with transversal gradation of effective properties
Rys. 2. Schemat struktury warstwy z poprzeczną gradacją własności efektywnych

MODELLING PROCEDURE

To functionally graded stratified composite, in the contrast to the periodic stratified structure is not assigned any representative sublayer. Instead of representative sublayer we introduce the basic concept of *the local (virtual) layer*. This is a formally defined sub-layer in new coordinate system $Oy^1x^2x^3$ occupying the region $\left(x - \frac{l}{2}, x + \frac{l}{2}\right) \times \Xi$ for an arbitrary, but fixed x ∈ (0, L). To this layer are uniquely assigned fractions $\varphi_A(x)$, $\varphi_B(x)$, such that $\varphi_A(x) + \varphi_B(x) = 1$. The scheme of this local (virtual) sub layer is shown in Figure 3. Obviously every virtual layer for $x \in \left\{\frac{l}{2}, \frac{l}{2} + l, \frac{l}{2} + 2l, ..., L - \frac{l}{2}\right\}$ coincides with the physical layer introduced in previous Section.

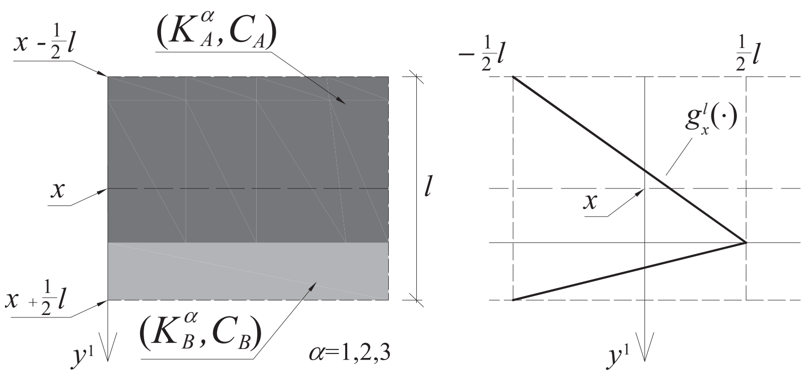

Fig. 3. A fragment of cross section of a local layer made of two homogenized materials
Rys. 3. Fragment przekroju warstwy lokalnej wykonanej z dwóch zhomogenizowanych materiałów

On the right hand side of Figure 3 there is also shown the diagram of what will be called the local shape function [cf. Woźniak et al. 2012]. The meaning of this function will be explained bellow.

Generally speaking the concept of local layer assigned to every $\left(x-\frac{l}{2}, x+\frac{l}{2}\right)\times \Xi$ is equivalent to every to the concept of the medium with microstructure in which to every point of the medium is assigned a certain but well defined new material structure; for the detailed study of this microstructure medium the reader is referred to Woźniak [1969].

The subsequent modelling procedure will be related to the concept of the arbitrary but fixed local layer. This procedure is based on two modelling assumptions. The first modelling assumption is strictly related to the weak limit passage.

Assumption 1 (Asymptotic approximation). The modelling procedure is realized under limit passage l → 0 independently for every l – periodic function $f_x^l \in L^2_{\text{loc}}(R)$, $x \in (0, L)$:

$$f_x^l \rightharpoonup \left\langle f_x^l \right\rangle \equiv \frac{1}{l}\int_{x-\frac{l}{2}}^{x+\frac{l}{2}} f_x(y)dy \text{ for every } x \in (0, L)$$

The second modelling assumption is referred to as the micro-macro decomposition.

Assumption 2 (Micro-macro decomposition of a local temperature field). Temperature field $\theta_x^l(\cdot)$ in every local layer is expected in the form:

$$\theta_x^l(\boldsymbol{y},t) = \vartheta(\boldsymbol{y},t) + g_x^l\left(y^1\right)\psi(\boldsymbol{y},t) \tag{4}$$

where: $g_x^l(\cdot)$ – a local shape function,

$\vartheta(\cdot), \psi(\cdot) \in C^1(\Omega\times[0,T])$ are the basic unknowns.

The micro-macro decomposition presented above makes it possible to satisfy the heat flux continuity on interfaces in the local layer; this continuity has to be satisfied only module $O(l)$. Bearing into account these assumptions we obtain the local averaged balance equation:

$$\partial_\alpha Q_x^\alpha + \langle C_x\rangle \partial_t \vartheta = 0 \tag{5}$$

where: $Q_x^1 \equiv -\left(\left\langle K_x^1\right\rangle + \gamma_x\left\langle K_x^1\partial_1 g_x\right\rangle\right)\partial_1\vartheta,$

$$Q_x^2 \equiv -\left\langle K_x^2\right\rangle\partial_2\vartheta,$$

$$Q_x^3 \equiv -\left\langle K_x^3\right\rangle\partial_3\vartheta,$$

$$\gamma_x = \left(K_A^1 - K_B^1\right)\left(\frac{K_A^1}{\varphi_A} + \frac{K_B^1}{\varphi_B}\right).$$

MODEL EQUATION

Taking into account the results outlined in previous Section we can define the following local effective coefficients:

$$\tilde{K}^1\left(x^1\right) \equiv \left\langle K^1_{x^1}\right\rangle + \gamma_{x^1}\left\langle K^1_{x^1}\partial_1 g_{x^1}\right\rangle = \frac{K^1_A K^1_B}{K^1_A\varphi_B\left(x^1\right) + K^1_B\varphi_A\left(x^1\right)}$$

$$\tilde{K}^2\left(x^1\right) \equiv \left\langle K^2_{x^1}\right\rangle = K^2_A\varphi_A\left(x^1\right) + K^2_B\varphi_B\left(x^1\right) \tag{6}$$

$$\tilde{K}^3\left(x^1\right) \equiv \left\langle K^3_{x^1}\right\rangle = K^3_A\varphi_A\left(x^1\right) + K^3_B\varphi_B\left(x^1\right)$$

$$\tilde{C}\left(x^1\right) \equiv \left\langle C_{x^1}\right\rangle = C_A\varphi_A\left(x^1\right) + C_B\varphi_B\left(x^1\right);\ x^1 \in (0, L)$$

Using the above effective coefficients, which are slowly varying functions depending on $x^1 \in (0, L)$ we obtain finally the heat balance equation in the form:

$$\partial_1\left(\tilde{K}^1\partial_1\vartheta\right) + \tilde{K}^2\partial_2\partial_2\vartheta + \tilde{K}^3\partial_3\partial_3\vartheta - \tilde{C}\dot{\vartheta} = 0 \tag{7}$$

Equation (7) together with formulas (6) for effective functional coefficients represent the final result of the proposed modelling procedure.

CONCLUSIONS

The proposed two step modelling procedure should lead to the reable results provided that that the compatibility condition $\xi << \min\limits_{n=1,\ldots,p} \varphi_A(x_n)l$ holds. This condition has to be satisfied with a sufficient accuracy which can not be verified in the framework of the proposing modelling procedure. In order to verify the scope of applicatibility of the above two step procedure it has to be evaluated the independent procedure which starts with a composite made of three-component materials "*a*", "*b*", "*R*".

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DWUETAPOWE ASYMPTOTYCZNE MODELOWANIE PRZEWODNICTWA CIEPŁA W WARSTWIE O FUNKCYJNEJ GRADACJI WŁASNOŚCI

Streszczenie. Celem artykułu jest przedstawienie dla warstwy o funkcyjnej gradacji własności równań przewodnictwa ciepła o wolnozmiennych współczynnikach. Zaproponowana procedura realizowania jest w dwóch etapach. W pierwszym etapie dwa różne izotropowe materiały wzmocnione są periodycznie rozmieszonymi bardzo cienkimi warstwami. Przy użyciu procedury homogenizacji otrzymuje się dwa nowe ortotropowe materiały. W drugim etapie te nowe materiały są połączone ze sobą i tworzą materiał o funkcyjnej gradacji własności, który jest podstawą dalszych analiz.

Słowa kluczowe: materiał typu FGM, poprzeczna gradacja efektywnych własności, przewodnictwo ciepła, homogenizacja

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