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# A NEW 2D-MODEL OF THE HEAT CONDUCTION IN MULTILAYERED MEDIUM-THICKNESS PLATES

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**Abstrakt.** 2D-models of multilayered plates are usually derived by a discretization approach across the plate thickness. Every discretized element coincides with a homogeneous layer of the medium and basic unknowns are assumed to be temperature and/or displacement fields on the plate interfaces. If the number of homogeneous layers is large then the discretization approach leads to a large number of basic unknowns. In this contribution there is proposed a new approach to the 2D-modelling of heat conduction which results in 2D-model equations for only two basic unknowns, independently of the number of layers.

**Keywords:** multilayered plate, heat conduction, 2D-modelling

## OBJECT OF ANALYSIS

The object of analysis is a plate which occupies region $\Omega \equiv \Pi \times \Delta,$ where $\Pi$ is a regular region on $Ox^1x^2$ plane, $\Delta \equiv \left(-\frac{\delta}{2}, \frac{\delta}{2}\right)$, where $\delta$ is the plate thickness. It is assumed that the plate thickness is small where compared to a minimum characteristic length dimension of the plane region $\Pi$; that is why the plate will be treated as a medium-thickness plate. Let interval $\Delta$ be divided into $N$ subintervals $(z_{n-1}, z_n)$ $n = 1, 2, ...., N,$ where: $z_0 = -\frac{\delta}{2}, z_N = \frac{\delta}{2}$. Regions $\Omega_n \equiv \Pi \times (z_{n-1}, z_n), \quad n = 1, 2, ..., N$ are assumed to be made of homogeneous and isotropic rigid heat conductors. It means that to every sublayer $\Omega_n \equiv \Pi \times (z_{n-1}, z_n), \quad n = 1, 2, ..., N$ there is assigned constant heat conduction factor $k_n$

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and constant specific heat $c_n$, $n = 1, 2, ..., N$. The thickness of the $n$-th sublayer will be denoted by $\delta_n$, $n = 1, 2, ..., N$; hence $\delta_1 + \delta_2 + ... + \delta_n = \delta$ and we shall denote $\varphi_n \equiv \frac{\delta_n}{\delta}$, $\varphi_1 + \varphi_2 + ... + \varphi_N = 1$. A fragment of cross section $x^2$ = const is shown in Figure 1.

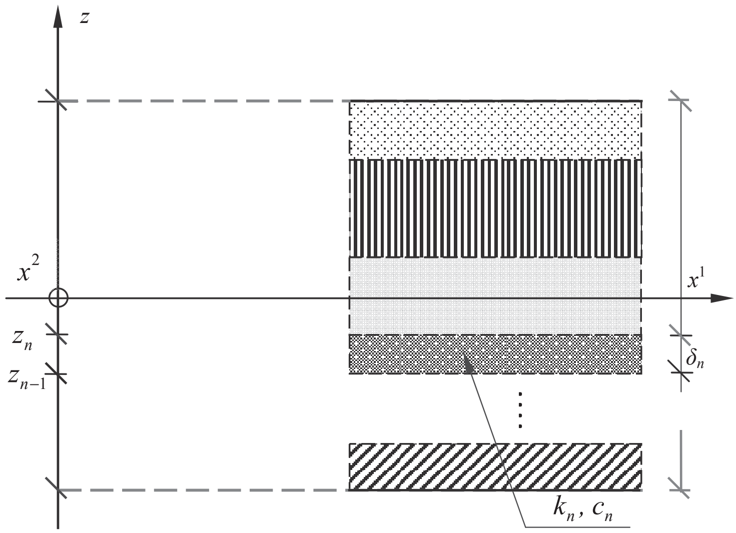


Fig. 1. A fragment of a cross section of the plate for $x^2$ = const

Setting $\mathrm{I} = \bigcup_{n=1}^{N-1} \{z_n\}$ we obtain $\Pi \times \mathrm{I}$ as a set of interfaces between homogeneous layers of the heat conductor. In the general case we shall assume that every pair of adjacent layers is made of different materials.

Functions $k(\cdot)$, $c(\cdot)$, defined on $\Delta - \mathrm{I}$ which attain constant values $k_n$, $c_n$, in every $(z_{n-1}, z_n)$ are assumed to determine uniquely all thermal properties of the plate under consideration.

The heat conduction in the plate under consideration will be described within the framework of the well known Fourier heat conduction theory. To this end denote by $\Theta(\boldsymbol{x}, z, t)$, $\boldsymbol{x} \equiv (x^1, x^2) \in \Pi$, $z \in \Delta$, $t \in [t_0, t_*)$, a continuous temperature field in $\Pi \times \Delta \times [t_o, t_*)$, $t$ is a time coordinate.

Define:

$$\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha}, \alpha = 1, 2; \ \overline{\nabla} \equiv (\partial_1, \partial_2); \ \partial \equiv \frac{\partial}{\partial z}, \dot{(\ )} \equiv \frac{\partial}{\partial t}, \boldsymbol{x} \equiv (x^1, x^2) \in \Pi, z \in \Delta, t \in [t_0, t_*)$$

$\partial^- f(z)$, $\partial^+ f(z)$ stand for left hand side and right hand side of the derivative of piece wise differentiable function respectively.

Let us denote by $f = f(\boldsymbol{x}, t)$, $\boldsymbol{x} \equiv (x^1, x^2) \in \Pi$, $t \in [t_0, t_*)$ the given *a priori* heat sources, representing the heat transported to the plate across the upper plate and lower plate boundary.

The heat balance equation inside the region occupied by plate is:

$$k(z)\overline{\nabla} \cdot \overline{\nabla}\Theta(\boldsymbol{x}, z, t) + \partial\left[k(z)\partial\Theta(\boldsymbol{x}, z, t)\right] - c(z)\dot{\Theta}(\boldsymbol{x}, z, t) = f(\boldsymbol{x}, t)$$

$$(\boldsymbol{x}, z) \in \Pi \times (\Delta - \mathrm{I}), \ t \in [t_0, t_*) \tag{1}$$

and has to be satisfied together with the heat flux continuity conditions across interfaces:

$$k(z_n^+)\,\partial^+\Theta(\boldsymbol{x}, z_n, t) = k(z_n^-)\,\partial^-\Theta(\boldsymbol{x}, z_n, t) \tag{2}$$

$$k(z_n^+) \equiv k_n \qquad \boldsymbol{x} \in \Pi, \ z_n \in \mathrm{I}$$

$$k(z_n^-) \equiv k_{n-1} \qquad t \in [t_0, t_*)$$

As well as the conditions on the upper and lower boundaries plate $z = \pm\delta / 2$.

In every initial-boundary value problem equations (1), (2) have to be considered together with the appropriate boundary and initial conditions.

## AIM OF CONTRIBUTION

The aim of contribution is to propose a certain new 2D- model of the heat conduction in the multilayered plate under consideration. To be more exact we represent the temperature field $\Theta(\boldsymbol{x}, z, t)$, $\boldsymbol{x} = (x^1, x^2) \in \Pi$, $z \in \left(-\frac{\delta}{2}, \frac{\delta}{2}\right)$ $t \in [t_0, t_*)$ in the form: $\Theta(\boldsymbol{x}, z, t) = \vartheta(\boldsymbol{x}, t) + [z + \gamma(z)]\psi(\boldsymbol{x}, t)$, where $\vartheta(\cdot), \psi(\cdot)$ are new unknowns and $\gamma(\cdot) \in C^0(\overline{\Delta})$ is postulated *a priori* function which will be specified bellow and is called the oscylating shape function.

At the same time we are to derive a system of partial differential equations with constant coefficients for aforementioned new unknowns under consideration. This system will be referred to as a 2D-model of the heat conduction in the plate under consideration. Obviously, a solution to a certain correctly stated initial-boundary value problem for $\vartheta(\cdot), \psi(\cdot)$ has to uniquely determine the temperature field, which should represent a sufficiently good approximation of the corresponding initial-boundary value problem for equations (1), (2).

The main difficulty of the above modelling procedure is that the functions $k(\cdot), c(\cdot)$ are discontinuous on interfaces. The problem of modelling of layered plates is not new. Among large number of references we shall mention here: Burmister [1945], Dong et al. [1962], Buffler [1971], Sun [1971], Woźniak Cz. [1978], Baczyński [2002], Baron [2002], Jędrysiak et al. [2006], and many others.

In most approaches the number of new unknowns and hence the number of 2D-model equations depends on the number $N$ of homogenous layers and is equal to $N$-1. This statement is usually related to the fact that the known 2D-models of multilayered plates are usually based on the discretization across the plate thickness into $N$ homogeneous sublayers.

For the approach proposed in this contribution the number of unknowns and the number of equations in the presented 2D-model is equal to 2 being independent of the number $N$ of homogenous sublayers. These unknowns are $\vartheta(\cdot)$, $\psi(\cdot)$.

## FUNDAMENTAL CONCEPT

The fundamental concept of the proposed approach is that of the oscillating shape function $\gamma(\cdot)$, which was introduced previously but not defined. We have stated above that this function is continuous and bounded in $\Delta$. We also postulate that:

(*i*) $\gamma(\cdot)$ is linear in every $(z_{n-1}, z_n)$ $n = 1, 2, ...., N$

(*ii*) function $\gamma(\cdot)$ satisfies boundary condition: $\gamma(z_0) = \gamma(z_N)$,

(*iii*) $\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \gamma(z)\, dz = 0$

(*iv*) values of function $\gamma(z_n)$, $n$ = 1, 2, ..., $N$ – 1 are given by the system of linear algebraic equations:

$$\frac{k_{n+1}}{\delta_{n+1}}(\gamma_{n+1} - \gamma_n) - \frac{k_n}{\delta_n}(\gamma_n - \gamma_{n-1}) = k_{n+1} - k_n \qquad n = 1, 2, ..., 2N - 1$$

## MODELLING HYPOTHESES

The proposed modelling approach is based on two hypotheses.

The first of them will be called *2D-modelling hypothesis* and states that the temperature field can be approximated by means of the formula:

$$\Theta(\boldsymbol{x}, z, t) = \vartheta(\boldsymbol{x}, t) + [z + \gamma(z)]\psi(\boldsymbol{x}, t) \tag{3}$$

$$\boldsymbol{x} = (x^1, x^2) \in \Pi, \ z \in \left(-\frac{\delta}{2}, \frac{\delta}{2}\right), \ t \in [t_0, t_*)$$

provided that the plate thickness is sufficiently small when compared to the smallest characteristic length dimension of the plane region $\Pi$.

The above restriction represents the necessary condition but is not sufficient.

The most important fact is that under decomposition (3) the heat flux continuity condition (2) is satisfied identically.

If we assume that on the lower and upper boundary plane the distribution of the temperatures for every $\boldsymbol{x} = (x^1, x^2) \in \Pi \quad t \in [t_0, t_*)$ are equal:

$$\Theta^{+}(\boldsymbol{x},t)=\vartheta(\boldsymbol{x},t)+\frac{\delta}{2}\psi(\boldsymbol{x},t) \quad \text{if} \quad z=\frac{\delta}{2}$$

$$\Theta^{-}(\boldsymbol{x},t)=\vartheta(\boldsymbol{x},t)-\frac{\delta}{2}\psi(\boldsymbol{x},t) \quad \text{if} \quad z=-\frac{\delta}{2}$$

Then the unknowns $\vartheta(\cdot)$ and $\psi(\cdot)$ are interpreted by the formulas:

$$\vartheta(\boldsymbol{x},t)=\frac{1}{2}\left(\Theta^{+}(\boldsymbol{x},t)+\Theta^{-}(\boldsymbol{x},t)\right)$$

$$\psi(\boldsymbol{x},t)=\frac{1}{\delta}\left(\Theta^{+}(\boldsymbol{x},t)-\Theta^{-}(\boldsymbol{x},t)\right)$$

Before the formulation of the second hypothesis we define the concept of the residual field defined on $\Pi\times\Delta$ and $t\in[t_0,t_*)$ and by means of:

$$r(\boldsymbol{x},z,t)=k(z)\overline{\nabla}\cdot\overline{\nabla}\Theta(\boldsymbol{x},z,t)-\partial\left[k(z)\partial\Theta(\boldsymbol{x},z,t)\right]-c(z)\dot{\Theta}(\boldsymbol{x},z,t)-f(\boldsymbol{x},t) \quad (4)$$

where in the right hand side of this formula the temperature field has to be substituted by equation (3).

The second 2D-modelling hypothesis is based on the well known definition of averaging which states that for every integrable function *F*(*z*) we define:

$$<F>\equiv\frac{1}{\delta}\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}}F(z)\,dz$$

Under the above denotation the second 2D-modelling hypothesis states that:

$$\begin{aligned}&<r>=0\\&<r(z+\gamma)>=0\end{aligned} \quad (5)$$

This is a specific case of the orthogonalization procedure.

## MODEL EQUATIONS

Realizing both hypothesis stated previously we obtain the following system of two partial differential equations with constant coefficients for unknowns $\vartheta(\cdot)$, $\psi(\cdot)$:

$$\begin{aligned}&\langle k(z)\rangle\overline{\nabla}\cdot\overline{\nabla}\vartheta(\boldsymbol{x},t)+\langle k(z)(z+\gamma(z))\rangle\overline{\nabla}\cdot\overline{\nabla}\psi(\boldsymbol{x},t)-\\&-\langle c(z)\rangle\dot{\vartheta}(\boldsymbol{x},t)-\langle c(z)(z+\gamma(z))\rangle\dot{\psi}(\boldsymbol{x},t)-f(\boldsymbol{x},t)=0\\&\langle k(z)(z+\gamma(z))\rangle\overline{\nabla}\cdot\overline{\nabla}\vartheta(\boldsymbol{x},t)+\left\langle k(z)(z+\gamma(z))^2\right\rangle\overline{\nabla}\cdot\overline{\nabla}\psi(\boldsymbol{x},t)-\\&-\langle c(z)(z+\gamma(z))\rangle\dot{\vartheta}(\boldsymbol{x},t)-\left\langle c(z)(z+\gamma(z))^2\right\rangle\dot{\psi}(\boldsymbol{x},t)-f(\boldsymbol{x},t)\langle(z+\gamma(z))\rangle=0\end{aligned} \quad (6)$$

Equations (6) represent the proposed 2D-model of the plate under consideration. It can be observed that for a homogeneous plate we have $N = 1$ and function $\gamma(\cdot)$ is identically equal to zero.

Now let us introduce the extra assumption that the plate midplane is a material symmetry plane. It means that functions $k(\cdot)$ and $c(\cdot)$ are even. From the aforementioned extra assumption and taking into account the definition of function $\gamma(\cdot)$ formulated in previous Section it follows that the function $\gamma(\cdot)$ is the odd function. The proof of this statement is rather simple. Equations (6) in this case reduce to the form:

$$\begin{aligned} &\langle k(z)\rangle \overline{\nabla}\cdot\overline{\nabla}\vartheta(\boldsymbol{x},t) - \langle c(z)\rangle \dot{\vartheta}(\boldsymbol{x},t) - f(\boldsymbol{x},t) = 0 \\ &\left\langle k(z)\left(z+\gamma(z)\right)^2\right\rangle \overline{\nabla}\cdot\overline{\nabla}\psi(\boldsymbol{x},t) - \left\langle c(z)\left(z+\gamma(z)\right)^2\right\rangle \dot{\psi}(\boldsymbol{x},t) = 0 \end{aligned} \tag{7}$$

Equations (7) are coupled only by means of the boundary and initial conditions for functions $\vartheta(\cdot)$ and $\psi(\cdot)$. This coupling is strictly related to the boundary conditions for temperature $\Theta(\boldsymbol{x}, z, t)$, $z \in (-\delta/2, \delta/2)$, $t \in [t_0, t_*)$ and initial conditions for temperature $\Theta(\boldsymbol{x}, z, t)$, $\boldsymbol{x} \in \Pi$, $z \in (-\delta/2, \delta/2)$, $t = t_0$.

## CONCLUSIONS AND REMARKS

Special example of 2-D model Equations will be restricted to $N = 3$. The oscillating shape function $\gamma(\cdot)$ for $N = 3$ is determined by the layer thicknesses $\delta_1$, $\delta_2$ and $\delta_3 = \delta_1$, by the heat conduction coefficients $k_1$, $k_2$ and $k_3 = k_1$. The scheme of the plate cross section and the diagram of oscillating shape function for $k_1 > k_2$ are shown in Figure 2.

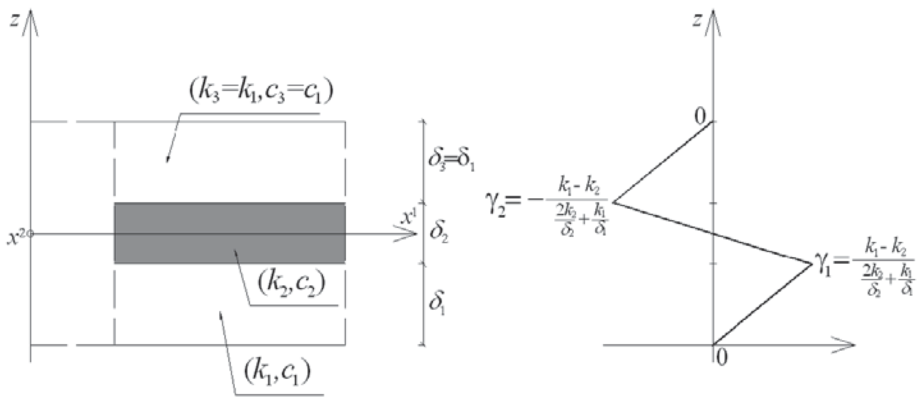


Fig. 2. The scheme of the plate cross section for $x^2$ = const and the diagram of oscillating shape function for $k_1 > k_2$

The corresponding model equations (7) are:

$$\begin{aligned}&\left\langle k(z)\right\rangle \overline{\nabla}\cdot\overline{\nabla}\vartheta(\boldsymbol{x},t)-\left\langle c(z)\right\rangle\dot{\vartheta}(\boldsymbol{x},t)-f(\boldsymbol{x},t)=0\\&\left\langle k(z)\left(z+\gamma(z)\right)^2\right\rangle\overline{\nabla}\cdot\overline{\nabla}\psi(\boldsymbol{x},t)-\left\langle c(z)\left(z+\gamma(z)\right)^2\right\rangle\dot{\psi}(\boldsymbol{x},t)=0\end{aligned}\tag{8}$$

We recall that the plate thickness $\delta = 2\delta_1 + \delta_2$ has to be sufficiently small with the smallest characteristic length dimension of the plate midplane $\Pi$. This requirement is necessary but not sufficient in applying the proposed 2D-model; this is a situation which is typical for any 2D-plate model which should appropriate the exact 3-D description of the thin plate.

The main advantage of the proposed 2-D model of the layered plate is evident if the number of homogeneous layers is large i.e. we deal with multilayered plates. We recall that the proposed model is represented by the system of only 2 partial differential equations for two unknown functions $\vartheta(\cdot), \psi(\cdot)$ independently to the number of layers.

In most approaches the number of new unknowns and hence the number of 2D-model equations depends on the number *N* of homogenous layers. This statement is usually related to the fact that the known 2D-models of multilayered plates are usually based on the discrietization across the plate thickness into *N* homogeneous sublayers.

At the end of this contribution let us take into account nontrivial situation in which the heat flux is directed exclusively along *Oz*-axis. In this case we obtain system of ordinary differential equations for $\vartheta(\cdot), \psi(\cdot)$ in the form:

$$\begin{aligned}&\left\langle c(z)\right\rangle\dot{\vartheta}(t)-f(t)=0\\&\left\langle c(z)\left(z+\gamma(z)\right)^2\right\rangle\dot{\psi}(t)=0\end{aligned}\tag{9}$$

From the formula of this system of equations it follows that if *f*(t) = 0 then $\vartheta(\cdot), \psi(\cdot)$ are constants.

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## 2D-MODEL PRZEWODNICTWA CIEPŁA W WIELOWARSTWOWYCH PŁYTACH ŚREDNIEJ GRUBOŚCI

**Streszczenie.** 2D-modele dla wielowarstwowych płyt są przeważnie uzyskiwane przez dyskretyzację wzdłuż grubości płyty. Każdy dyskretyzowany element jest jednorodną warstwą, z której składa się płyta. Podstawowymi niewiadomymi są powstałe w wyniku procesu dyskretyzacji: pole temperatury i/lub pole przemieszczenia na powierzchniach rozgraniczających poszczególne warstwy. Jeśli liczba jednorodnych warstw jest duża, wtedy podejście dyskretyzacyjne prowadzi do dużej liczby podstawowych niewiadomych. W prezentowanej pracy zostało zaproponowane nowe podejście do modelowania przewodnictwa ciepła, w którego wyniku otrzymujemy nowy 2D-model z dwiema niewiadomymi, niezależnie od liczby jednorodnych warstw, z których zbudowana jest płyta.

**Słowa kluczowe:** wielowarstwowa płyta, przewodnictwo ciepła, 2D-modelowanie

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