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ON THE THERMAL BOUNDARY EFFECT BEHAVIOR IN THE HEXAGONAL-TYPE BIPERIODIC ISOTROPIC DIVIDING WALLS

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Abstract. This paper shows the analysis of the temperature boundary effect behavior in the dividing wall made of the conductor with bi-periodic material structure in which every surface parallel to the outside and the inside surface is bi-periodic. Conductor is made of a special case of hexagonal-type material structure. As a tool of modeling there has been chosen the tolerance averaging technique.

Key words: tolerance averaging, heat conduction, biperiodic conductors, boundary effect

INTRODUCTION. CASE DEFINITION

This paper deals situations in which there is necessity to protect the interior of some space from highly oscillating external temperatures. It shows some attempt of using the tolerance averaging technique [Woźniak and Wierzbicki 2000, Woźniak (ed.) 2009, 2010, Jędrysiak 2010] to consider heat conduction in wall consist of biperiodic hexagonal-type material structure (Fig. 1.) that would have properties described below. Issues of modeling of hexagonal structures has been already raised in [Woźniak and Wierzbicki 2000, Nagórko and Wągrowska 2002, Cielecka and Jędrysiak 2006].

The aim of this paper is to analyze some special kind of behavior observed in considered structures that is called boundary effect behavior. This phenomena consist on suppressing the fluctuation amplitudes in very thin selvedge layer of considered conductor. The boundary effect is described by boundary effect equation.

First, let's introduce the mentioned hexagonal type structure. In the analyzed case every cell is divided into three material rhombus parts $\lambda\Diamond_1, \lambda\Diamond_2, \lambda\Diamond_3$ with constant thermal properties which in that case are isotropic. Single hexagon will be named as *basic cell* and denoted by $\lambda\Delta$. λ is a side length of such cell and it is equal to 1.

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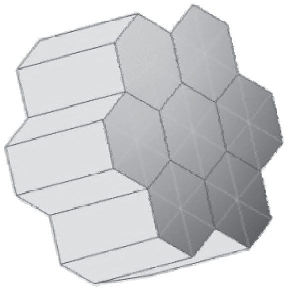

Fig. 1. The conductor with hexagonal-type material structure
Rys. 1. Przewodnik o strukturze heksagonalnej

$\lambda\Delta$ is illustrated into Figure 2. Let's also pay attention to the three vectors $t^1 = [1, 0, 0]^T$, $t^2 = [-0.5,\ \sqrt{3}/2,\ 0]^T$, $t^3 = [-0.5,\ -\sqrt{3}/2,\ 0]^T$ on Figure 2. These three vectors coincide with shorter diagonals of three rhombus and their introduction is necessary to analyze problems of heat conduction in next paragraphs.

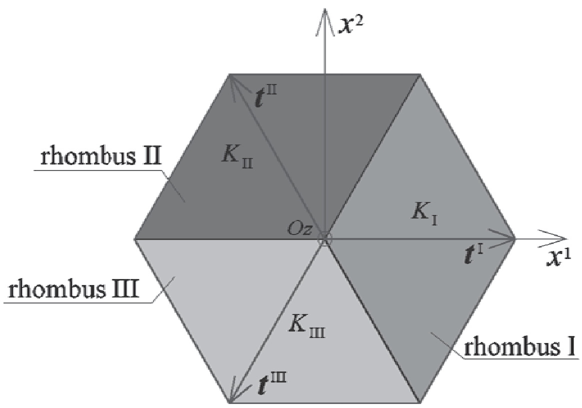

Fig. 2. Basic hexagonal cell
Rys. 2. Podstawowa komórka heksagonalna

The hexagonal cell is situated in Carthesian orthogonal coordinate system $Ox^1x^2x^3$ in which the plane Ox^1x^2 is a biperiodicity plane, axis Ox^1 includes the shorter diagonal of a distinguish rhombus with the first number and $Ox^3 = Oz$ is normal to mentioned biperiodicity plane.

The considerations of this paper are limited to special case in which thermal properties of each rhombus are described by heat conductivity tensor:

$$K = K(x) = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}, \ \kappa \equiv [k_{13}, k_{23}]^T \tag{1}$$

and by specific heat $c(\cdot)$. It has to be emphasize that in considered case the heat conductivity tensor $K(\cdot)$ is proportional to identity matrix and for each of three rhombus we get:

$$K_a = k^a I = k^a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad a = I, II, III \tag{2}$$

Also matrix $K_a = k^a I$ is the 2 x 2 heat conductivity matrix and it is created after removing the third row and third column from K_a.

For the purpose of further consideration there is assumed jump discontinuous of $K(\cdot) \in R^{3\times 3}$ on interfaces between three rhombus sections which are treated as perfectly bonded [Vutz and Angrist 1970, Sideman and Moalem-Maron 1982].

BOUNDARY EFFECT EQUATION

Boundary effect equation is a part of one of the model equations received using the tolerance averaging technique, cf. [Woźniak and Wierzbicki 2000, Nagórko 2008, Michalak 2010], from the well-known parabolic heat transfer equation:

$$c\dot{\theta} - (\nabla + \partial) \cdot [K(\nabla + \partial)]\theta = b \tag{3}$$

Symbol b means the heat sources field, c is specific heat field and θ is temperature field. We also denote $\nabla \equiv \mathrm{grad} + \partial$ for $\mathrm{grad} \equiv [\partial_1, \partial_2, 0]^T$, $\partial \equiv [0, 0, \partial_3]^T$, $\partial_1 \equiv \partial / \partial x^1$, $\partial_2 \equiv \partial / \partial x^2$, $\partial_3 \equiv \partial / \partial x^3$. Assuming that:

$$\boldsymbol{v} = \boldsymbol{t}^s \psi_s, \quad s = 1,\ 2,\ 3 \tag{4}$$

where ψ_S are fluctuation amplitudes and

$$\mathbf{v} = \mathrm{e}^{-\frac{2H}{3\langle g k_{33} g\rangle}\frac{x_3}{\lambda}} \mathbf{w} \tag{5}$$

we get two forms of boundary effect equation:

$$\lambda^2[\langle gcg\rangle \dot{\mathbf{v}} - \langle g k_{33} g\rangle \partial_3 \partial_3 \mathbf{v})] - \frac{4}{3}\lambda \mathrm{H} \partial_3 \mathbf{v} + \{\mathrm{K}\}\mathbf{v} = 0 \tag{6}$$

and

$$\lambda^2[\langle gcg\rangle \dot{\mathbf{w}} - \langle g k_{33} g\rangle \partial^T \partial \mathbf{w}] + \mathrm{M}(\mathrm{x}_3)\mathbf{w} = 0 \tag{7}$$

Here and in the sequel symbol $\langle\cdot\rangle$ stands for integral averaging operation over related *representative cell*, which in this contribution will be identified as $2\lambda\Delta$. Vector will be referred to as *a fluctuation vector* and its two first coordinates are the same asin **v**. The third coordinate is equal to 0 so it can be skipped and we can use two-coordinate vector **v**. The concept of the fluctuation vector has been already used in the tolerance modeling approach, cf. [Kula et al. 2012, Mazewska and Wierzbicki 2012, 2013a, b]. Vector **w** also has two coordinates and is referred to as a *generalized amplitude vector*. Exponential coefficient in (5) is treated as a value of the Lapunov exponent operator on matrix $-\frac{2H}{3\langle gk_{33}g\rangle}\frac{x_3}{\lambda}$, cf. [Lai-Sang 2013]. The coefficients that appear in both equations (6) and (7) in considered case are described by:

$$
\begin{aligned}
\{\mathrm{K}\} &= \frac{1}{4}[(\mu^1)^2(k^{III}+k^{II})+(\mu^2)^2(k^{I}+k^{III})+(\mu^3)^2(k^{II}+k^{I})](1+o(\varepsilon))\ \mathrm{I} \\
\langle gk_{33}g\rangle &= (k_{33}) = \frac{1}{2}[(\mu^1)^2 k_{33}^{I}+(\mu^2)^2 k_{33}^{II}+(\mu^3)^2 k_{33}^{III}]\langle gg\rangle(1+o(\varepsilon)) \\
\mathrm{H} &= 0
\end{aligned} \tag{8}
$$

μ are parameters, that will be described in next paragraph. I is 2 x 2 identity matrix. Coefficient M = M(x_3) is here a matrix that appears after changing **v** into **w**:

$$
\mathrm{M}(x_3) = \mathrm{e}^{\frac{2\mathrm{H}}{3\langle gk_{33}g\rangle}\frac{x_3}{\lambda}}\{\mathrm{K}\}\mathrm{e}^{-\frac{2\mathrm{H}}{3\langle gk_{33}g\rangle}\frac{x_3}{\lambda}} - \frac{4\mathrm{H}^2}{9\langle gk_{33}g\rangle} \overset{\mathrm{H}=0}{=} \{\mathrm{K}\} \tag{9}
$$

All new symbols used in (8) will be presented below. First we need to focus on the function $g(x)$. We deal with three of this functions. Those functions are some parts of $h^1(x), h^2(x), h^3(x)$ defined by $h^r(x) = \lambda g^r(\lambda^{-1}x)$, for $x \in \mathrm{R}^2$ and r = 1, 2, 3. The choice of $h^1(x), h^2(x), h^3(x)$ will be realized as:

$$
\begin{aligned}
h^1(x) &= \lambda[\mu^1 g_{\mathrm{peaks}}(\lambda^{-1}x)+\varepsilon d^1 g_{\mathrm{mound}}(\lambda^{-1}x)] \\
h^2(x) &= \lambda[\mu^2 g_{\mathrm{peaks}}(\lambda^{-1}\mathrm{rot}_Q(x))+\varepsilon d^2 g_{\mathrm{mound}}(\lambda^{-1}\mathrm{rot}_Q(x)) \\
h^3(x) &= \lambda[\mu^3 g_{\mathrm{peaks}}(\lambda^{-1}\mathrm{rot}_Q\circ\mathrm{rot}_Q(x))+\varepsilon d^3 g_{\mathrm{mound}}(\lambda^{-1}\mathrm{rot}_Q\circ\mathrm{rot}_Q(x))]
\end{aligned} \tag{10}
$$

where $\mathrm{rot}_Q(x) = Q(x - x_0)^T + x_0$, $x \in \mathrm{R}^2$ means rotation over 2π/3 in R^2 with an arbitrary chosen origin of hexagonal cell as a center of rotation and Q is orthogonal matrix of rotation over 2π/3 in R^2:

$$
\mathrm{Q} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \tag{11}
$$

Residual parts $\varepsilon\lambda d^r g_{\mathrm{mound}}(\lambda^{-1}\mathrm{rot}_Q{}^{r-1}(x))$ tends to zero while $\varepsilon \to 0$.

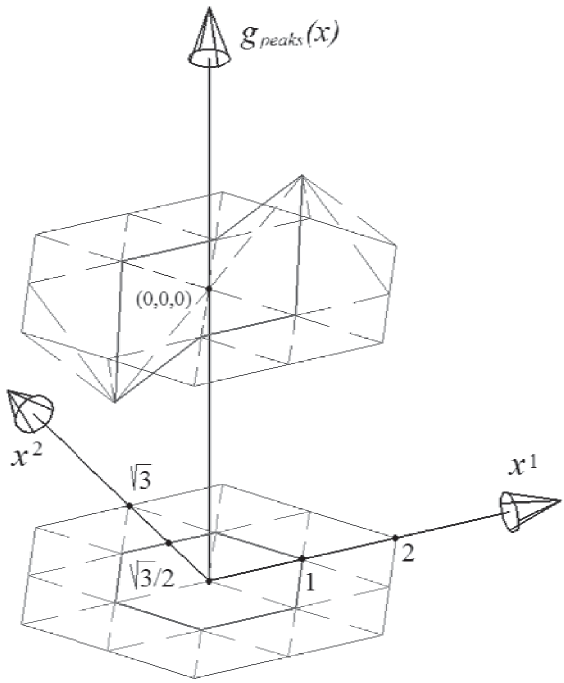

Fig. 3. Function g_{peaks}
Rys. 3. Funkcja g_{peaks}

This residuals, together with six parameters μ^1, μ^2, μ^3, d^1, d^2, d^3 should result a requirement that related continuity conditions imposed the tolerance heat flux vector should be satisfied.

Under introduced functions $h^1(x)$, $h^2(x)$, $h^3(x)$ continuity condition (for the normal component of the tolerance heat flux) takes place if the following two equations are satisfied. First refers to parameters μ^1, μ^2, μ^3:

$$\begin{bmatrix} k^{III}-k^{II} & k^{III}-k^{I} & 0 \\ 0 & k^{I}-k^{III} & k^{II}-k^{I} \\ k^{III}-k^{II} & 0 & k^{II}-k^{I} \end{bmatrix} \begin{bmatrix} \mu^1 \\ \mu^2 \\ \mu^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{12}$$

and the second to parameters d^1, d^2, d^3:

$$\begin{bmatrix} 0 & k^{II} & k^{III} \\ k^{I} & 0 & k^{III} \\ k^{I} & k^{II} & 0 \end{bmatrix} \begin{bmatrix} d^1 \\ d^2 \\ d^3 \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} \tag{13}$$

where:

$$D_1 = (1+o(\varepsilon))\{((\mu^2)^2+(\mu^3)^2)k^I+((\mu^3)^2+(\mu^1)^2)k^{II}+((\mu^1)^2+(\mu^2)^2)k^{III}\}/$$
$$/(\mu^3(k^{II}-k^I))-\mu^1(k^{II}-k^{III}),$$
$$D_2 = (1+o(\varepsilon))\{((\mu^2)^2+(\mu^3)^2)k^I+((\mu^3)^2+(\mu^1)^2)k^{II}+((\mu^1)^2+(\mu^2)^2)k^{III}\}/$$
$$/(\mu^1(k^{III}-k^{II}))-\mu^2(k^{III}-k^I),$$
$$D_3 = (1+o(\varepsilon))\{((\mu^2)^2+(\mu^3)^2)k^I+((\mu^3)^2+(\mu^1)^2)k^{II}+((\mu^1)^2+(\mu^2)^2)k^{III}\}/$$
$$/(\mu^2(k^I-k^{III}))-\mu^3(k^I-k^{II})$$

Solution to equation (12) is not unique and can be written in the form:

$$\begin{cases} \mu^1 = \dfrac{(k^I-k^{II})\zeta}{k^{III}-k^{II}} \\ \mu^2 = \dfrac{(k^I-k^{II})\zeta}{k^I-k^{III}} \\ \mu^3 = \zeta \end{cases} \tag{14}$$

in which $\zeta \in R$. As the ε goes to 0, the part marked as $o(\varepsilon)$ satisfy condition $o(\varepsilon) << 1$ and hence it can be ignored in expressions (13) as well as in (8).

ILLUSTRATIVE PROBLEM

Boundary effect problem for considered issue is related to the stationary case of boundary effect equation (7) which in such case takes form:

$$\lambda^2 \langle g k_{33} g \rangle \partial^T \partial \mathbf{w} - \mathrm{M}\mathbf{w} = 0 \tag{15}$$

together with boundary conditions:

$$\begin{gathered} \mathbf{w}(z = x_3 = 0) = \mathbf{w}_0 \\ \mathbf{w}(z = x_3 = L) = \mathbf{w}_L \\ \{\mathbf{w}_0, \mathbf{w}_L\} = \begin{bmatrix} \mathrm{w}_1 \\ \mathrm{w}_2 \end{bmatrix} \end{gathered} \tag{16}$$

We will use solution to the similar simpler problem formulated and investigated in Woźniak et al. [2002] which can be rewritten in the form:

$$\mathbf{w} = \mathrm{e}^{-\frac{1}{\lambda}\sqrt{\chi}x_3}\left(\frac{\mathrm{e}^{2\frac{1}{\eta}\sqrt{\chi}}}{\mathrm{e}^{2\frac{1}{\eta}\sqrt{\chi}}-1}\mathbf{w}_0 - \frac{\mathrm{e}^{\frac{1}{\eta}\sqrt{\chi}}}{\mathrm{e}^{2\frac{1}{\eta}\sqrt{\chi}}-1}\mathbf{w}_L\right) +$$

$$+\mathrm{e}^{\frac{1}{\lambda}\sqrt{\chi}x_3}\left(-\frac{1}{\mathrm{e}^{2\frac{1}{\eta}\sqrt{\chi}}-1}\mathbf{w}_0 + \frac{\mathrm{e}^{\frac{1}{\eta}\sqrt{\chi}}}{\mathrm{e}^{2\frac{1}{\eta}\sqrt{\chi}}-1}\mathbf{w}_L\right) \tag{17}$$

In above $\eta = \lambda / L$ and $\chi = \frac{\mathrm{M}}{(k_{33})}$ which in considered case is equivalent to $\chi = \frac{\{\mathrm{K}\}}{(k_{33})}$.

Boundary conditions can be also rewritten for temperatures θ and u as it is shown in above Figure 4. The connection between constant temperatures $\theta_{\mathrm{outside}}$ and θ_{inside} and averaged temperatures u_{outside} and u_{inside} is called micro-macro hypothesis which is well known and frequently used in tolerance averaging technique.

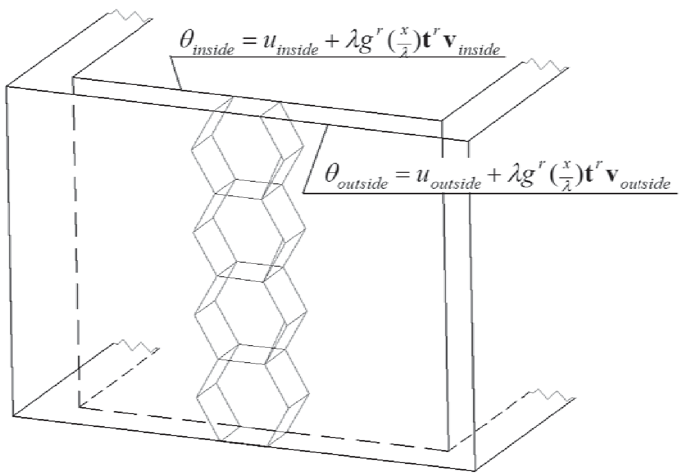

Fig. 4. Hexagonally periodic dividing wall
Rys. 4. Przegroda o periodycznej strukturze heksagonalnej

To show some solution of boundary effect equation (15) there have been assumed some values of k^I, k^{II}, k^{III}:

$$k^{I} = k_{33}^{I} = 0.29 \ \frac{\mathrm{W}}{\mathrm{m{\cdot}K}}$$
$$k^{II} = k_{33}^{II} = 0.77 \ \frac{\mathrm{W}}{\mathrm{m{\cdot}K}} \qquad (18)$$
$$k^{III} = k_{33}^{III} = 1.00 \ \frac{\mathrm{W}}{\mathrm{m{\cdot}K}}$$

There have been calculated proper values of parameters μ^r and d^r:

$$\begin{cases} \mu^1 = -0.31 \\ \mu^2 = 0.10 \\ \mu^3 = 0.15 \end{cases} \Rightarrow \begin{cases} d^1 = -15.00 \\ d^2 = 1.95 \\ d^3 = 1.35 \end{cases} \qquad (19)$$

Under introduced values of parameters in (19) continuity condition (for the normal component of the tolerance heat flux) is satisfied. Since we have assumed that averaged values of g_{mouds} (x^1, x^2) over the area of whole representative cell are close to zero, to calculate the value of coefficient (k_{33}) there should be taken only function g_{peaks} (x_1, x_2). For $g_{\text{peaks}}(x_1, x_2)$ we have $\langle g_{\text{peaks}} g_{\text{peaks}} \rangle = \langle gg \rangle = 0.125$. For proposed k^{I}, k^{II}, k^{III} we get $\chi = 14.25$. Moreover, $\lambda = 0.02$ m or 0.1 m and width of wall $L = 0.3$ m and hence $\eta = 0.07$ or $\eta = 0.33$, respectively.

There have been also assumed values of boundary conditions (16):

$$\mathbf{w}_0 = \begin{bmatrix} 5 \\ 5 \end{bmatrix}, \ \mathbf{w}_L = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad (20)$$

Both boundary vectors $\mathbf{w}_0$ and $\mathbf{w}_L$ satisfy condition $\mathrm{w}_1 = \mathrm{w}_2$ and hence the graphs for w_1 and w_2 coincide.

The quotient $\sqrt{\chi} / \eta$ is treated as certain measure of the intensity of boundary effect behavior. The below graphs illustrate the interrelation between intensity of boundary effect behavior $\sqrt{\chi} / \eta$ and nondimentional microstructure parameter η for three arbitrary fixed values of χ and mean conductivity parameter χ for three arbitrary fixed values of η.

FINAL REMARKS

The interpretation of solution showed on Figure 5 is related with results presented on Figure 6 and Figure 7. The intensity of boundary effect grows with increase of mean conductivity parameter χ and decrease of nondimensional microstructure parameter η. Boundary effect is stronger for smaller cell and larger width of wall.

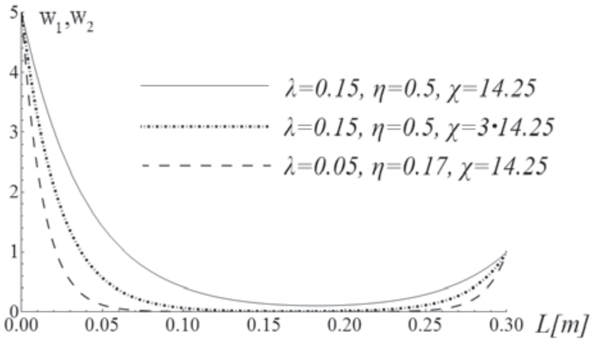

Fig. 5. Solution to boundary effect issue
Rys. 5. Rozwiązanie zagadnienia efektu brzegowego

Three lines presented on Figure 5 shows that:

1. The most important for boundary effect intensity is small dimension of cell λ and small value of quotient λ/L.

2. It is better to reduce the dimension of cell than to increase the value of χ which can be received by improvement of thermal properties of wall in the third direction (parallel to axis *Oz*). Examples on Figure 5 shows that stronger boundary effect appears for three times smaller cell than for three times higher value χ (compare dashed and dot-dashed lines).

3. The wall should have weaker thermal properties in the surface of biperiodicity and better in the third direction (parallel to axis *Oz*).

4. For sufficiently large dimension of cell boundary effect intensity is small or even completely disappears (compare dashed and continuous lines).

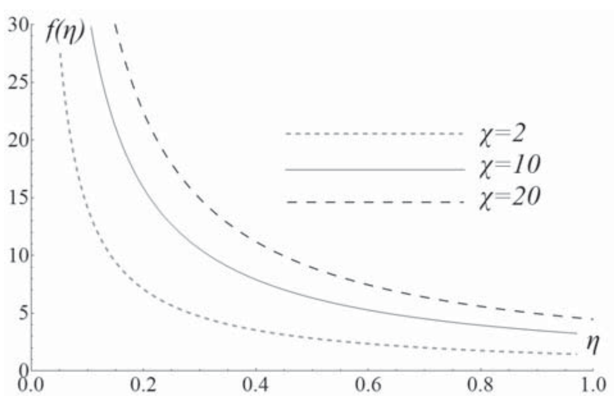

Fig. 6. The interrelation between intensity of boundary effect behavior and nondimensional microstructure parameter
Rys. 6. Zależność między efektem brzegowym a bezwymiarowym parametrem mikrostruktury

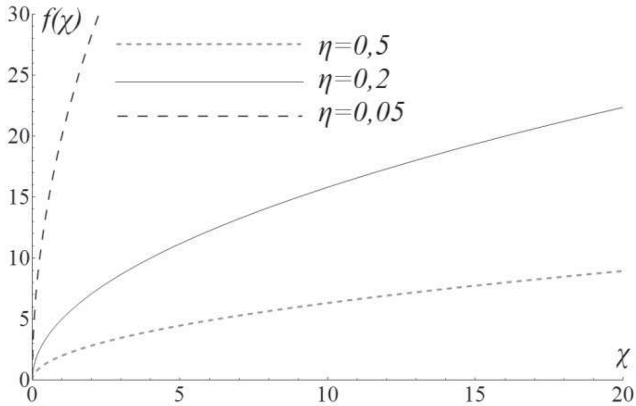

Fig. 7. The interrelation between intensity of boundary effect behavior and mean conductivity parameter
Rys. 7. Zależność między efektem brzegowym a parametrem przewodnictwa cieplnego

This paper considers heat conductivity for some special case of hexagonal-type material structure in which the single hexagonal cell consists of three constituents with rhombus cross-section. Constituents have different but isotropic material structure. A new shape function proposed in the framework assures that tolerance heat flux vector has continuous crossing normal to the surfaces between constituents of hexagonal cell. The most important results received in this paper concern boundary effect problem solution for described hexagonal-type dividing wall.

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ZJAWISKO EFEKTU BRZEGOWEGO W DWUKIERUNKOWO PERIODYCZNEJ PRZEGRODZIE O IZOTROPOWEJ STRUKTURZE HEKSAGONALNEJ

Streszczenie. W niniejszej pracy przeanalizowano zjawisko temperaturowego efektu brzegowego w przegrodzie wykonanej z przewodnika o dwukierunkowo periodycznej strukturze materialnej, w której każda powierzchnia równoległa do wewnętrznej i zewnętrznej powierzchni jest płaszczyzną periodyczności. Przewodnik wykonany jest ze szczególnego rodzaju struktury heksagonalnej. Jako narzędzie badań wykorzystano technikę tolerancyjnego uśredniania.

Słowa kluczowe: uśrednianie tolerancyjne, przewodnictwo ciepła, przewodniki dwukierunkowo periodyczne, efekt brzegowy

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