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STABILITY OF ANISOTROPIC CYLINDRICAL SHELLS IN THREE-DIMENSIONAL STATE UNDER AXIAL COMPRESSION

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Abstract. The paper presents an approach to the solution of a problem of stability of cylindrical anisotropic shells under the influence of axial compression. The approach is based on the application of the Bubnov-Galerkin procedure, taking into account boundary conditions on surfaces and end edges of cylindrical shells as well as on the application of a numerical method of discrete orthogonalization. It has been solved a problem of stability of cylindrical shells made of a material characterized by one plane of elastic symmetry. It has been investigated a dependence of values of critical stresses on the rotation angle of main directions of elasticity properties in respect to the main curvatures of the cylindrical shell. The results are presented in form of graphs and provided in the tables. Their analysis also has been carried out.

Key words: stability, cylindrical anisotropic shells, Bubnov-Galerkin procedure, three-dimensional state

INTRODUCTION

Extensive literature is dedicated to the problem of calculations of stability of thin, thick and medium thickness shells, made of isotropic or orthotropic materials [Volmir 1967, Ambartsumyan 1974, Richards and Teters 1974, Guz' and Babich 1985, Guz' 1986, Vanin and Semenyuk 1987]. Some of researchers [Korolyev 1965, Mykyshcheva 1968, Mytkevich and Kul'kov 2006] classify the shells made out of fibrous composites wound on a mandrel as orthotropic. The material of such structures is orthotropic if related to

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its own axes. However, after winding this material becomes anisotropic and gains properties of a material with a single plane of elastic symmetry. Appearance of such type of anisotropy is due to a mismatch of the directions of orthotropy of the fibrous material and the main directions of coordinate lines of a given shell structure. Interconnection in the elasticity ratios between tension and shift, tension and bend, shift and torsion is the factor that, the most frequently, causes a meaningful reduction in critical stresses for anisotropic shell structures. It is confirmed by many researchers: Weaver et al. [2002], Weaver [2003], Semenyuk and Trach [2004, 2006], Trach and Podvornyi [2004], Wong and Weaver [2005], Trach [2006, 2007, 2008], Bazhenov et al. [2010], Weaver [2015], Kolakowski and Teter [2015].

The topic on stability of thin anisotropic shells has been researched by many researchers and the results were presented in publications by Korolyev [1965], Karmyshyn et al. [1975], Guz' [1986], Grygoryenko and Kryukov [1988]. The most thorough analysis is carried out in the monograph by Bazhenov et al. [2010].

However, many questions on determination of stability parameters of medium thickness anisotropic shell constructions still require further clarification. Therefore, such calculations are still topical and are not only of scientific but also of practical importance. It is commonly known that the calculations of such constructions should be conducted with the use of specified theories. Works of many researchers have been dedicated to solve this problem: Vanin and Semenyuk [1987], Reddy [2015], Weawer [2015], Kolakowski and Teter [2015], Muc and Pastuszko [2015], Trach and Khoruzhyy [2015], Castro et al. [2015]. The authors of the works cited above used the simplest and most well-known refined Timoshenko's theory that is based on the kinematic hypothesis.

Computational problems concerning stress-strain state and stability of anisotropic composite thick shell constructions are the least researched because of the complexities in creating analytical and numerical methods of calculations of such three-dimensional systems. However, it is noteworthy that such solutions exist for cylindrical thick shells. Monographs by Guz' and Babich [1985], Guz' [1986] describe calculations of stability of cylindrical shells being under the influence of axial stresses and external pressures. The researches by J. Grygorenko, A. Vasylenko, N. Pankratova present methods of description of a stress-strain state of a thick shell under the influence of external pressure by means of the introduction of appropriate types of boundary conditions on its surface.

This paper describes the researches on the stability of thick anisotropic composite cylindrical shells, made of a material with one plane of elastic symmetry, under the influence of axial compression.

DESCRIPTION OF THE PROBLEM

Let us consider an elastic cylindrical shell in a cylindrical coordinate system r, z, θ. Coordinates z and θ match with the principal curvature lines of the construction, r– normal coordinate or cylinder radius that does not depend on the coordinates z and θ. The material anisotropy is described by the angle of rotation of the main directions of elasticity of the material relatively to the z axis in the assumed coordinate system (Fig. 1).

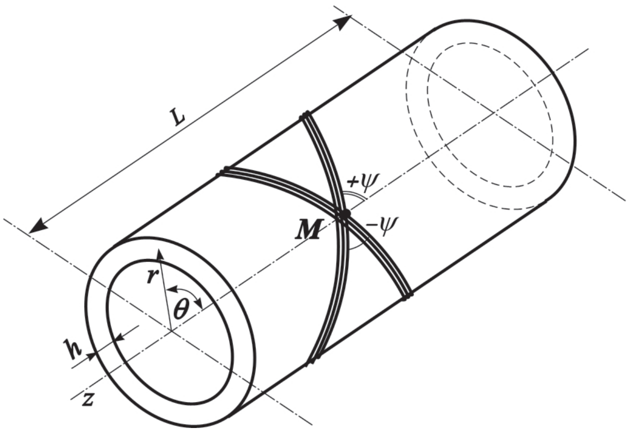

Fig. 1. Cylindrical thick anisotropic shell

The equations of equilibrium written below are based on the work of Novozhylov [1948]:

$$\frac{\partial \hat{\sigma}_{rr}}{\partial r} = -\frac{1}{r}\left[\hat{\sigma}_{rr} + r\frac{\partial}{\partial z}\left(\hat{\tau}_{zr}\right) + \frac{\partial}{\partial \theta}\left(\hat{\tau}_{\theta r}\right) - \hat{\sigma}_{\theta\theta} + rF_r\right]$$

$$\frac{\partial \hat{\tau}_{rz}}{\partial r} = -\frac{1}{r}\left[\hat{\tau}_{rz} + r\frac{\partial}{\partial z}\left(\hat{\sigma}_{zz}\right) + \frac{\partial}{\partial \theta}\left(\hat{\tau}_{\theta z}\right) + rF_z\right] \tag{1}$$

$$\frac{\partial \hat{\tau}_{r\theta}}{\partial r} = -\frac{1}{r}\left[\hat{\tau}_{r\theta} + \hat{\tau}_{\theta r} + r\frac{\partial}{\partial z}\left(\hat{\tau}_{z\theta}\right) + \frac{\partial}{\partial \theta}\left(\hat{\sigma}_{\theta\theta}\right) + rF_\theta\right]$$

where F_r, F_z, F_θ – projections of the volume force vectors on the directions tangent to the coordinate lines r, z, θ.

Normal and tangent stresses are projected in the form of Novozhylov [1948]:

$$\hat{\sigma}_{rr} = \sigma_{rr} + \sigma_{rr}\frac{\partial u_r}{\partial r} + \tau_{rz}\frac{\partial u_r}{\partial z} + \tau_{r\theta}\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \tau_{r\theta}\frac{1}{r}u_\theta$$

$$\hat{\sigma}_{zz} = \sigma_{zz} + \sigma_{zz}\frac{\partial u_z}{\partial z} + \tau_{z\theta}\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \tau_{rz}\frac{\partial u_z}{\partial r}$$

$$\hat{\sigma}_{\theta\theta} = \sigma_{\theta\theta} + \sigma_{\theta\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \sigma_{\theta\theta}\frac{1}{r}u_r + \tau_{z\theta}\frac{\partial u_\theta}{\partial z} + \tau_{r\theta}\frac{\partial u_\theta}{\partial r} \tag{2}$$

$$\hat{\tau}_{rz} = \tau_{rz} + \tau_{rz}\frac{\partial u_z}{\partial z} + \sigma_{rr}\frac{\partial u_z}{\partial r} + \tau_{r\theta}\frac{1}{r}\frac{\partial u_z}{\partial \theta}$$

$$\hat{\tau}_{zr} = \tau_{rz} + \tau_{rz}\frac{\partial u_r}{\partial r} + \sigma_{zz}\frac{\partial u_r}{\partial z} + \tau_{z\theta}\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \tau_{z\theta}\frac{1}{r}u_\theta$$

$$\hat{\tau}_{r\theta} = \tau_{r\theta} + \tau_{r\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \tau_{r\theta}\frac{1}{r}u_r + \sigma_{rr}\frac{\partial u_\theta}{\partial r} + \tau_{rz}\frac{\partial u_\theta}{\partial z}$$

$$\hat{\tau}_{\theta r} = \tau_{r\theta} + \tau_{r\theta}\frac{\partial u_r}{\partial r} + \tau_{z\theta}\frac{\partial u_r}{\partial z} + \sigma_{\theta\theta}\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \sigma_{\theta\theta}\frac{1}{r}u_\theta \tag{2}$$

$$\hat{\tau}_{z\theta} = \tau_{z\theta} + \tau_{z\theta}\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \tau_{z\theta}\frac{1}{r}u_r + \sigma_{zz}\frac{\partial u_\theta}{\partial z} + \tau_{rz}\frac{\partial u_\theta}{\partial r}$$

$$\hat{\tau}_{\theta z} = \tau_{z\theta} + \tau_{z\theta}\frac{\partial u_z}{\partial z} + \tau_{r\theta}\frac{\partial u_z}{\partial r} + \sigma_{\theta\theta}\frac{1}{r}\frac{\partial u_z}{\partial \theta}$$

The relationship between the components of the elasticity and displacements has the form given by Novozhylov [1948]:

$$e_{zz} = \frac{\partial u_z}{\partial z}; \quad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r}u_r; \quad e_{rr} = \frac{\partial u_r}{\partial r};$$

$$e_{z\theta} = \frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}; \quad e_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}; \quad e_{r\theta} = \frac{\partial u_\theta}{\partial r} - \frac{1}{r}u_\theta + \frac{1}{r}\frac{\partial u_r}{\partial \theta} \tag{3}$$

where: u_z, u_θ, u_r – displacement of the cylinder points in the direction of axes z, θ, r respectively.

The relations of the generalized Hooke Law, connecting the components of deformation and stress after rotation of the orthotropy axes relatively to the z axis, are written below in the form given by Lekhnytskyy [1977]:

$$e_{zz} = a_{11}\sigma_{zz} + a_{12}\sigma_{\theta\theta} + a_{13}\sigma_{rr} + a_{16}\tau_{z\theta}$$

$$e_{\theta\theta} = a_{12}\sigma_{zz} + a_{22}\sigma_{\theta\theta} + a_{23}\sigma_{rr} + a_{26}\tau_{z\theta}$$

$$e_{rr} = a_{13}\sigma_{zz} + a_{23}\sigma_{\theta\theta} + a_{33}\sigma_{rr} + a_{36}\tau_{z\theta}$$

$$e_{r\theta} = a_{44}\tau_{r\theta} + a_{45}\tau_{rz} \tag{4}$$

$$e_{rz} = a_{45}\tau_{r\theta} + a_{55}\tau_{rz}$$

$$e_{z\theta} = a_{16}\sigma_{zz} + a_{26}\sigma_{\theta\theta} + a_{36}\sigma_{rr} + a_{66}\tau_{z\theta}$$

In equation (4), $a_{ij}\,(i, j = \overline{1,6})$ are mechanical constants of a material with one plane of elastic symmetry. Following Lekhnytskyy [1977] the 13 constants in the directions of main curvatures of the cylinder can be expressed by 9 orthotropic constants in the main elasticity axes:

$$a_{11} = a'_{11}\cos^4\psi + \left(2a'_{12} + a'_{66}\right)\cos^2\psi\sin^2\psi + a'_{22}\sin^4\psi$$

$$a_{22} = a'_{22}\cos^4\psi + \left(2a'_{12} + a'_{66}\right)\cos^2\psi\sin^2\psi + a'_{11}\sin^4\psi$$

$$a_{12} = a'_{12} + \left(a'_{11} + a'_{22} - 2a'_{12} - a'_{66}\right)\sin^2\psi\cos^2\psi$$

$$a_{66} = a'_{66} + 4\left(a'_{11} + a'_{22} - 2a'_{12} - a'_{66}\right)\cos^2\psi\sin^2\psi$$

$$a_{16} = \left[2a'_{22}\sin^2\psi - 2a'_{11}\cos^2\psi + \left(2a'_{12} + a'_{66}\right)\left(\cos^2\psi - \sin^2\psi\right)\right]\cos\psi\sin\psi$$

$$a_{26} = \left[2a'_{22}\cos^2\psi - 2a'_{11}\sin^2\psi - \left(2a'_{12} + a'_{66}\right)\left(\cos^2\psi - \sin^2\psi\right)\right]\cos\psi\sin\psi \tag{5}$$

$$a_{13} = a'_{13}\cos^2\psi + a'_{23}\sin^2\psi$$

$$a_{23} = a'_{23}\cos^2\psi + a'_{13}\sin^2\psi$$

$$a_{36} = 2\left(a'_{23} - a'_{13}\right)\cos\psi\sin\psi$$

$$a_{33} = a'_{33}$$

$$a_{44} = a'_{44}\cos^2\psi + a'_{55}\sin^2\psi$$

$$a_{55} = a'_{55}\cos^2\psi + a'_{44}\sin^2\psi$$

$$a_{45} = \left(a'_{44} - a'_{55}\right)\cos\psi\sin\psi$$

where: ψ – angle of rotation of the main direction of elasticity of an orthotropic material in relation to the z-axis of the cylindrical coordinates and the quantities a'_{ij} $(i, j = \overline{1,6})$ are defined by the dependences derived by Grygoryenko et al. [1991].

METHODOLOGY OF SOLUTION OF THE PROBLEM

The relationships of the generalized Hooke Law for materials with one plane of elastic symmetry (4) have been transformed to the following form and used to solve the system of equations (1):

$$\sigma_{zz} = b_{11}e_{zz} + b_{12}e_{\theta\theta} + b_{16}e_{z\theta} + c_1\sigma_{rr}$$

$$\sigma_{\theta\theta} = b_{12}e_{zz} + b_{22}e_{\theta\theta} + b_{26}e_{z\theta} + c_2\sigma_{rr} \tag{6}$$

$$\begin{aligned}
\tau_{z\theta} &= b_{16}e_{zz} + b_{26}e_{\theta\theta} + b_{66}e_{z\theta} + c_3\sigma_{rr} \\
e_{rr} &= -c_1 e_{zz} - c_2 e_{\theta\theta} - c_3 e_{z\theta} + c_4\sigma_{rr} \\
e_{rz} &= a_{45}\tau_{r\theta} + a_{55}\tau_{rz} \\
e_{r\theta} &= a_{44}\tau_{r\theta} + a_{45}\tau_{rz}
\end{aligned} \tag{6}$$

where: $b_{ij}\,(i, j = 1,\,2,\,6)$, $c_i\,(i = \overline{1,4})$ – characteristics calculated with the use of the mechanical constants $a_{ij}\ (i, j = \overline{1,3};5;6)$ of the shell material [Grygoryenko et al. 1991].

Equations of stability, based on the Euler's static criterion, are derived from the system (1), taking the dependencies (2), (3) into account:

$$\begin{aligned}
\frac{\partial\sigma_{rr}}{\partial r} &= -\frac{1}{r}\left[\sigma_{rr} + r\frac{\partial}{\partial z}\left(\tau_{zr} + \sigma_{zz}^0\left(\frac{\partial u_r}{\partial z}\right) + \tau_{z\theta}^0\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} - \frac{1}{r}u_\theta\right)\right) + \frac{\partial}{\partial\theta}\left(\tau_{\theta r} + \tau_{z\theta}^0\left(\frac{\partial u_r}{\partial z}\right)\right) - \right. \\
&\quad \left. -\left(\sigma_{\theta\theta} + \tau_{z\theta}^0\left(\frac{\partial u_\theta}{\partial z}\right)\right)\right] \\
\frac{\partial\tau_{rz}}{\partial r} &= -\frac{1}{r}\left[\tau_{rz} + r\frac{\partial}{\partial z}\left(\sigma_{zz} + \sigma_{zz}^0\frac{\partial u_z}{\partial z} + \tau_{z\theta}^0\left(\frac{1}{r}\frac{\partial u_z}{\partial\theta}\right)\right) + \frac{\partial}{\partial\theta}\left(\tau_{z\theta} + \tau_{z\theta}^0\frac{\partial u_z}{\partial z}\right)\right] \\
\frac{\partial\tau_{r\theta}}{\partial r} &= -\frac{1}{r}\left[\tau_{r\theta} + \left(\tau_{r\theta} + \tau_{z\theta}^0\frac{\partial u_r}{\partial z}\right) + r\frac{\partial}{\partial z}\left(\tau_{z\theta} + \tau_{z\theta}^0\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{1}{r}u_r\right) + \sigma_{zz}^0\frac{\partial u_\theta}{\partial z}\right) + \right. \\
&\quad \left. + \frac{\partial}{\partial\theta}\left(\sigma_{\theta\theta} + \tau_{z\theta}^0\frac{\partial u_\theta}{\partial z}\right)\right]
\end{aligned} \tag{7}$$

where: σ_{zz}^0 and $\tau_{z\theta}^0$ are the known subcritical stress values. Since the problem of axial compression is considered here, the prevailing stresses are σ_{zz}^0 and $\tau_{z\theta}^0$, what is proved by the results of the researches on the stress state of cylindrical shells made of orthotropic materials and materials with one plane of elastic symmetry [Lekhnytskyy 1977, Grygoryenko et al. 1991]. Subcritical stress values are represented in the system (7). It shows the heterogeneity of the subcritical stress state of the cylindrical shell.

A relationship between these stresses has been derived for such a construction. Since until the moment when the stability is lost the shell preserves its non-deformed state, then in subcritical state of deformation $e_{z\theta}$ equals to zero and is described as (4):

$$e_{z\theta} = a_{16}\sigma_{zz}^0 + a_{66}\tau_{z\theta}^0 = 0$$

where only the prevailing subcritical stresses in axial compression are taken into account.

Based on this, one can identify a relationship between the axial and tangential subcritical stresses:

$$\tau_{z\theta}^{0} = -\frac{a_{16}}{a_{66}}\sigma_{zz}^{0} \tag{8}$$

Replacement of the deformations e_{zz}, $e_{\theta\theta}$, $e_{z\theta}$, e_{rz}, $e_{r\theta}$, e_{rr} in equation (6) with their expressions from equation (3) and substitution of the resulting dependencies σ_{zz}, $\sigma_{\theta\theta}$, $\tau_{z\theta}$ into equation (7) yields the following system of equations of stability:

$$\frac{\partial \sigma_{rr}}{\partial r} = \frac{c_2 - 1}{r}\sigma_{rr} - \frac{\partial \tau_{rz}}{\partial z} - \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{b_{22}}{r^2}u_r + \frac{b_{12}}{r}\frac{\partial u_z}{\partial z} + \frac{b_{26}}{r^2}\frac{\partial u_z}{\partial \theta} + \frac{b_{26}}{r}\frac{\partial u_\theta}{\partial z} + \frac{b_{22}}{r^2}\frac{\partial u_\theta}{\partial \theta} -$$

$$-\sigma_{zz}^{0}\frac{\partial^2 u_r}{\partial z^2} - \frac{2}{r}\tau_{z\theta}^{0}\frac{\partial^2 u_r}{\partial z \partial \theta} + \frac{2}{r}\tau_{z\theta}^{0}\frac{\partial u_\theta}{\partial z}$$

$$\frac{\partial \tau_{rz}}{\partial r} = -c_1\frac{\partial \sigma_{rr}}{\partial z} - \frac{1}{r}\tau_{rz} - \frac{b_{12}}{r}\frac{\partial u_r}{\partial z} - b_{11}\frac{\partial^2 u_z}{\partial z^2} - \frac{b_{66}}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} - \frac{b_{12}+b_{66}}{r}\frac{\partial^2 u_\theta}{\partial z \partial \theta} - \frac{c_3}{r}\frac{\partial \sigma_{rr}}{\partial \theta} -$$

$$-\frac{b_{26}}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{2b_{16}}{r}\frac{\partial^2 u_z}{\partial z \partial \theta} - b_{16}\frac{\partial^2 u_\theta}{\partial z^2} - \frac{b_{26}}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} - \sigma_{zz}^{0}\frac{\partial^2 u_z}{\partial z^2} - \frac{2}{r}\tau_{z\theta}^{0}\frac{\partial^2 u_z}{\partial z \partial \theta} \tag{9}$$

$$\frac{\partial \tau_{r\theta}}{\partial r} = -\frac{c_2}{r}\frac{\partial \sigma_{rr}}{\partial \theta} - \frac{2}{r}\tau_{r\theta} - \frac{b_{22}}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{b_{12}+b_{66}}{r}\frac{\partial^2 u_z}{\partial z \partial \theta} - b_{66}\frac{\partial^2 u_\theta}{\partial z^2} - \frac{b_{22}}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} - c_3\frac{\partial \sigma_{rr}}{\partial z} -$$

$$-\frac{b_{26}}{r}\frac{\partial u_r}{\partial z} - b_{16}\frac{\partial^2 u_z}{\partial z^2} - \frac{b_{26}}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} - \frac{2b_{26}}{r}\frac{\partial^2 u_\theta}{\partial z \partial \theta} - \sigma_{zz}^{0}\frac{\partial^2 u_\theta}{\partial z^2} - \frac{2}{r}\tau_{z\theta}^{0}\frac{\partial u_r}{\partial z} - \frac{2}{r}\tau_{z\theta}^{0}\frac{\partial^2 u_\theta}{\partial z \partial \theta}$$

$$\frac{\partial u_r}{\partial r} = c_4\sigma_{rr} - \frac{c_2}{r}u_r - c_1\frac{\partial u_z}{\partial z} - \frac{c_3}{r}\frac{\partial u_z}{\partial \theta} - c_3\frac{\partial u_\theta}{\partial z} - \frac{c_2}{r}\frac{\partial u_\theta}{\partial \theta}$$

$$\frac{\partial u_z}{\partial r} = a_{55}\tau_{rz} + a_{45}\tau_{r\theta} - \frac{\partial u_r}{\partial z}$$

$$\frac{\partial u_\theta}{\partial r} = a_{45}\tau_{rz} + a_{44}\tau_{r\theta} - \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{1}{r}u_\theta$$

To solve equation (9), one can assume any value of σ_{zz}^{0} and then, according to equation (8), determine the value $\tau_{z\theta}^{0}$.

Solution of the system (9) has been found with assumption of boundary conditions:

- on the inner surface ($r = r_0$) and the outer surface ($r = r_n$) of the shell

$$\sigma_{rr} = \tau_{rz} = \tau_{r\theta} = 0 \tag{10}$$

– on the end edges

$$\sigma_{zz} = -\sigma_{zz}^{0}, \quad u_r = u_\theta = 0 \tag{11}$$

That can prove the existence of absolutely rigid diaphragms on the end edges of the cylinder and flexible conditions on its upper and lower surfaces.

Hereinafter the Bubnov-Galerkin procedure was used to transform the three-dimensional problem into one-dimensional one which was solved below with the application of the method of discrete orthogonalization. In accordance with this, all functions have been developed into trigonometric series according to the coordinates along the generatrix z so that all of them satisfy the boundary conditions (11), and their periodicity has been also taken into account by considering a frequency θ:

$$\begin{aligned}
\sigma_{rr}(r,z,\theta) &= \sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\left[y_{1,pk}(r)\cos k\theta + y'_{1,mk}(r)\sin k\theta\right]\sin l_m z \\
\tau_{rz}(r,z,\theta) &= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\left[y_{2,pk}(r)\cos k\theta + y'_{2,mk}(r)\sin k\theta\right]\cos l_m z \\
\tau_{r\theta}(r,z,\theta) &= \sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\left[y_{3,pk}(r)\sin k\theta + y'_{3,mk}(r)\cos k\theta\right]\sin l_m z \\
u_r(r,z,\theta) &= \sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\left[y_{4,pk}(r)\cos k\theta + y'_{4,mk}(r)\sin k\theta\right]\sin l_m z \\
u_z(r,z,\theta) &= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\left[y_{5,pk}(r)\cos k\theta + y'_{5,mk}(r)\sin k\theta\right]\cos l_m z \\
u_\theta(r,z,\theta) &= \sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\left[y_{6,pk}(r)\sin k\theta + y'_{6,mk}(r)\cos k\theta\right]\sin l_m z
\end{aligned} \tag{12}$$

After several mathematical transformations and the separation of the variables in the equations (9) with help of the relations (12), it has been derived an infinite system of ordinary differential equations of stability in the Cauchy's normal form:

$$\frac{d\bar{y}}{dr} = T(r)\bar{y}, \quad T(r) = t_{i,j}(r) \tag{13}$$

where: $\bar{y} = \left\{y_{1,pk}; y_{2,pk}; y_{3,pk}; y_{4,pk}; y_{5,pk}; y_{6,pk}; y'_{1,mk}; y'_{2,mk}; y'_{3,mk}; y'_{4,mk}; y'_{5,mk}; y'_{6,mk}\right\}$ – solving vector function. The non-zero elements of the matrix $T(r)$ are shown below:

$$t_{1,1} = \frac{c_2 - 1}{r}, \quad t_{1,2} = l_p, \quad t_{1,3} = -\frac{k}{r}, \quad t_{1,4} = \frac{b_{22}}{r^2} + \sigma_{zz}^{0} l_p^2, \quad t_{1,5} = -l_p \frac{b_{12}}{r} \quad t_{1,6} = k\frac{b_{22}}{r^2},$$

$$t_{1,10} = \sum_{m=1}^{\infty} \varphi(p,m) 2\frac{k}{r}\tau_{z\theta}^{0} l_m, \quad t_{1,11} = \sum_{m=1}^{\infty}\varphi(p,m) k \frac{b_{26}}{r^2},$$

$$t_{1,12}=\sum_{m=1}^{\infty}\varphi(p,m)\left(\frac{b_{26}}{r}l_m+\frac{2}{r}\tau_{z\theta}^0 l_m\right),\quad t_{2,1}=-c_1 l_p,\quad t_{2,2}=-\frac{1}{r},\quad t_{2,4}=-\frac{b_{12}}{r}l_p,$$

$$t_{2,5}=b_{11}l_p^2+k^2\frac{b_{66}}{r^2}+\sigma_{zz}^0 l_p^2,\quad t_{2,6}=-k\frac{b_{12}+b_{66}}{r}l_p,\quad t_{2,7}=-\sum_{m=0}^{\infty}\varphi(m,p)k\frac{c_3}{r},$$

$$t_{2,10}=-\sum_{m=0}^{\infty}\varphi(m,p)k\frac{b_{26}}{r^2},\quad t_{2,11}=\sum_{m=0}^{\infty}\varphi(m,p)\left(2\frac{kb_{16}}{r}l_m+2\frac{k}{r}\tau_{z\theta}^0 l_m\right),$$

$$t_{2,12}=\sum_{m=0}^{\infty}\varphi(m,p)\left(b_{16}l_m^2+k^2\frac{b_{26}}{r^2}\right),\quad t_{3,1}=k\frac{c_2}{r},\quad t_{3,3}=-\frac{2}{r},\quad t_{3,4}=k\frac{b_{22}}{r^2},$$

$$t_{3,5}=-k\frac{b_{12}+b_{66}}{r}l_p,\quad t_{3,6}=b_{66}l_p^2+k^2\frac{b_{22}}{r^2}+\sigma_{zz}^0 l_p^2,\quad t_{3,7}=-\sum_{m=1}^{\infty}\varphi(p,m)c_3 l_m,$$

$$t_{3,10}=-\sum_{m=1}^{\infty}\varphi(p,m)\left(\frac{b_{26}}{r}l_m+\frac{2}{r}\tau_{z\theta}^0 l_m\right),\quad t_{3,11}=\sum_{m=1}^{\infty}\varphi(p,m)\left(b_{16}l_m^2+k^2\frac{b_{26}}{r^2}\right),$$

$$t_{3,12}=\sum_{m=1}^{\infty}\varphi(p,m)\left(2\frac{kb_{26}}{r}l_m+2\frac{k}{r}\tau_{z\theta}^0 l_m\right),\quad t_{4,1}=c_4,\quad t_{4,4}=-\frac{c_2}{r},\quad t_{4,5}=c_1 l_p,$$

$$t_{4,6}=-k\frac{c_2}{r},\quad t_{4,11}=-\sum_{m=1}^{\infty}\varphi(p,m)k\frac{c_3}{r},\quad t_{4,12}=-\sum_{m=1}^{\infty}\varphi(p,m)c_3 l_m,\quad t_{5,2}=a_{55},$$

$$t_{5,4}=-l_p,\quad t_{5,9}=\sum_{m=0}^{\infty}\varphi(m,p)a_{45},\quad t_{6,3}=a_{44},\quad t_{6,4}=\frac{k}{r},\quad t_{6,6}=\frac{1}{r},$$

$$t_{6,8}=\sum_{m=1}^{\infty}\varphi(p,m)a_{45},\quad t_{7,4}=-\sum_{m=1}^{\infty}\varphi(p,m)2\frac{k}{r}\tau_{z\theta}^0 l_m,\quad t_{7,5}=\sum_{m=1}^{\infty}\varphi(p,m)k\frac{b_{26}}{r^2},$$

$$t_{7,6}=\sum_{m=1}^{\infty}\varphi(p,m)\left(\frac{b_{26}}{r}l_m+\frac{2}{r}\tau_{z\theta}^0 l_m\right),\quad t_{7,7}=\frac{c_2-1}{r},\quad t_{7,8}=l_p,\quad t_{7,9}=-k\frac{1}{r},$$

$$t_{7,10}=\frac{b_{22}}{r^2}+\sigma_{zz}^0 l_p^2,\quad t_{7,11}=-l_p\frac{b_{12}}{r},\quad t_{7,12}=k\frac{b_{22}}{r^2},\quad t_{8,1}=-\sum_{m=0}^{\infty}\varphi(m,p)k\frac{c_3}{r},$$

$$t_{8,4}=-\sum_{m=0}^{\infty}\varphi(m,p)k\frac{b_{26}}{r^2},\quad t_{8,5}=\sum_{m=0}^{\infty}\varphi(m,p)\left(2\frac{kb_{16}}{r}l_m+2\frac{k}{r}\tau_{z\theta}^0 l_m\right),$$

$$t_{8,6}=\sum_{m=0}^{\infty}\varphi(m,p)\left(b_{16}l_m^2+k^2\frac{b_{26}}{r^2}\right),\quad t_{8,7}=-c_1 l_p,\quad t_{8,8}=-\frac{1}{r},\quad t_{8,10}=-\frac{b_{12}}{r}l_p,$$

$$t_{8,11}=b_{11}l_p^2+k^2\frac{b_{66}}{r^2}+\sigma_{zz}^0 l_p^2,\quad t_{8,12}=-k\frac{b_{12}+b_{66}}{r}l_p,\quad t_{9,1}=-\sum_{m=1}^{\infty}\varphi(p,m)c_3 l_m,$$

$$t_{9,4} = -\sum_{m=1}^{\infty} \varphi(p,m)\left(\frac{b_{26}}{r} l_m + \frac{2}{r}\tau_{z\theta}^0 l_m\right),\ \ t_{9,5} = \sum_{m=1}^{\infty} \varphi(p,m)\left(b_{16} l_m^2 + k^2 \frac{b_{26}}{r^2}\right),$$

$$t_{9,6} = \sum_{m=1}^{\infty} \varphi(p,m)\left(2\frac{kb_{26}}{r} l_m + 2\frac{k}{r}\tau_{z\theta}^0 l_m\right),\ \ t_{9,7} = -k\frac{c_2}{r},\ \ t_{9,9} = -\frac{2}{r},\ \ t_{9,10} = k\frac{b_{22}}{r^2},$$

$$t_{9,11} = -k\frac{b_{12}+b_{66}}{r} l_p,\ \ t_{9,12} = b_{66} l_p^2 + k^2\frac{b_{22}}{r^2},\ \ t_{10,5} = -\sum_{m=1}^{\infty} \varphi(p,m) k\frac{c_3}{r}, \tag{14}$$

$$t_{10,6} = -\sum_{m=1}^{\infty} \varphi(p,m) c_3 l_m,\ \ t_{10,7} = c_4,\ \ t_{10,10} = -\frac{c_2}{r},\ \ t_{10,11} = c_1 l_p,\ \ t_{10,12} = -k\frac{c_2}{r},$$

$$t_{11,3} = \sum_{m=0}^{\infty} \varphi(m,p) a_{45},\ \ t_{11,8} = a_{55},\ \ t_{11,10} = -l_p,\ \ t_{12,2} = \sum_{m=1}^{\infty} \varphi(p,m) a_{45},$$

$$t_{12,9} = a_{44},\ \ t_{12,10} = \frac{k}{r},\ \ t_{12,12} = \frac{1}{r}$$

where: $l_m = \frac{m\pi}{L}$, $l_p = \frac{p\pi}{L}$, L – length of the cylinder generatrix, p, m – wave numbers in the Fourier series (12).

The functions $\varphi(p, m)$ and $\varphi(m, p)$ depend on the integral numerical parameters p, m and are determined from the formulas:

$$\varphi(p,m) = \begin{cases} 0, & \text{if } p+m \text{ – even number,} \\ \frac{2}{\pi}\left(\frac{1}{p-m} + \frac{1}{p+m}\right) & \text{if } p+m \text{ – odd number.} \end{cases}$$
$$\varphi(m,p) = \begin{cases} 0, & \text{if } p+m \text{ – even number,} \\ \frac{2}{\pi}\left(\frac{1}{m-p} + \frac{1}{m+p}\right) & \text{if } p+m \text{ – odd number.} \end{cases} \tag{15}$$

Implementation of the resulting system of equations (13) with the given boundary conditions (10) has been conducted with the use of the numerical method of discrete orthogonalization [Grygoryenko and Kryukov 1988]. The solving algorithm for the problem of stability of axially compressed shells of revolution is developed as a software package.

One of the approaches confirming reliability of the obtained critical values of the axial compression are convergence tests of the solution of the stability problem by increasing in the members of a trigonometric row. As the performed calculations have shown, the critical values stopped to change when already five members of a row (12) were used.

RESULTS OF THE NUMERICAL CALCULATIONS AND THEIR ANALYSIS

Tests of the results of the solution based on the recommended approach of stability of cylindrical orthotropic shells under the influence of axial compression have been carried out and then compared to the values of the critical stresses obtained by Guz' and Babich [1985]. The shell having been calculated has: radius R = 0.6 m, length L = 2.15 m, physical and mechanical material properties $E_{11} = 10.0E_0$, $E_{22} = 2.8E_0$, $E_{33} = E_0$, $G_{12} = 1.075E_0$, $G_{13} = G_{23} = 2E_0$, $v_{21} = 0.3$, $v_{12} = 0.084$, $v_{32} = 0.22$, $v_{31} = 0.35$, E_0 = 1,0 MN·m^2. The comparison of the obtained values of critical stresses is presented in Table 1.

The analysis of the results of the test calculations of stability, shown in Table 1, indicates the full conformity of the solutions obtained according to the recommended approach and the results presented in the work of Guz' and Babich [1985]. Unfortunately, the authors of the present article were not able to find in the literature any reliable data concerning calculations of the stability of anisotropic cylindrical shells in a three-dimensional system.

Table 1. Results of the test calculations of stability

Wall thickness	Calculation results based on methodology acc. Guz' and Babich		Calculation results based on recommended methodology	
h [m]	number of waves	critical stress value σ_{cr} [MN·m^2]	number of waves	critical stress value σ_{cr} [MN·m^2]
0.012	6	$4.0 \cdot 10^4$	6	$4.0 \cdot 10^4$
0.02	5	$6.5 \cdot 10^4$	5	$6.5 \cdot 10^4$
0.025	4	$8.0 \cdot 10^4$	4	$8.0 \cdot 10^4$

As an example of application of this methodology, let us consider the stability of a cylindrical shell of constant thickness built of multiple cross-stacked layers of a fibrous material, laid at the angles $\pm\psi_i$ to the axis z. The shell parameters correspond to those of the previous problem and the wall thickness h is assumed as 0.03 m. Eight types of shells under axial pressure have been tested. The constructions under consideration differed by the number of cross-stacked layers, changing from one to eight. For the single layered orthotropic shell, the calculation has been carried out without taking anisotropic material constants a_{i6}, $i = \overline{1,3}$, into account.

The values of the critical buckling stresses shown in Table 2 should be multiplied by 10^2 MN·m^2, wherein near each of the critical values the number of waves of stability loss is presented in the brackets. Figure 2 displays the results of the calculations, respectively for the shells with: 1 – one layer (+ψ), 2 – two layers, cross-wound (+–ψ), 3 – three layers (+–+ψ), 4 – four layers (+–+–ψ); 5 – shells calculated without taking anisotropic constants of material into account. Results of the calculations of other types of shells are not displayed for sake of a drawing's clarity.

The character of changes in the critical stresses, shown in Table 2 and graphically presented in Figure 2, suggests that their values under the axial pressure highly depend on the value $\pm\psi_i$ of the stacked layers whose quantity varies from one to five. Furthermore, an increase in number of shell layers does not have any impact on the values of the critical axial stresses, approaching critical values obtained without taking anisotropic material constants into account.

Table 2. Stability of a cylindrical shell of constant thickness built of multiple cross-stacked layers

Winding angle ψ, Degrees	Single-layered shell	Double-layered (cross-wound shell)	Triple-layered (cross-wound shell)	Quadruple-layered (cross-wound shell)	Quintuple-layered (cross-wound shell)	Septuple-layered (cross-wound shell)	Octuple-layered (cross-wound shell)	Calculations without taking anisotropic material constants into account
	F_{cr}	F_{cr}	F_{cr}	F_{cr}	F_{cr}	F_{cr}	F_{cr}	F_{cr}
0	28.2 (4)	28.2 (4)	28.2 (4)	28.2 (4)	28.2 (4)	28.2 (4)	28.2 (4)	28.2(4)
10	20.0 (3)	27.4 (4)	28.5 (4)	30.1 (4)	31.0 (3)	31.0 (3)	30.9 (4)	31.0(4)
20	16.3 (3)	28.2 (3)	28.8 (3)	32.6 (3)	32.8 (3)	32.8 (3)	32.7 (3)	32.8(3)
30	15.9 (3)	28.5 (2)	29.6 (3)	33.2 (3)	33.6 (3)	33.5 (3)	33.4 (3)	33.5(3)
40	18.6 (3)	28.8 (1)	31.7 (1)	31.9 (0)	32.6 (1)	32.8 (1)	32.6 (0)	32.7(0)
50	24.8 (2)	30.7 (0)	32.1 (1)	32.1 (0)	32.5 (0)	32.6 (0)	32.6 (0)	32.6(0)
60	27.6 (2)	30.0 (3)	32.3 (2)	31.4 (2)	31.4 (2)	31.5 (2)	31.6 (2)	31.6(2)
70	27.4 (4)	27.5 (4)	28.4 (4)	28.3 (2)	29.8 (2)	29.8 (2)	29.3 (2)	29.9(2)
80	26.6 (4)	26.3 (4)	27.4 (4)	27.4 (2)	28.0 (2)	28.1 (2)	28.0 (2)	28.2(2)
90	28.0 (4)	28.0 (4)	28.0 (4)	28.0 (4)	28.0 (4)	28.0 (4)	28.0 (4)	28.0(4)

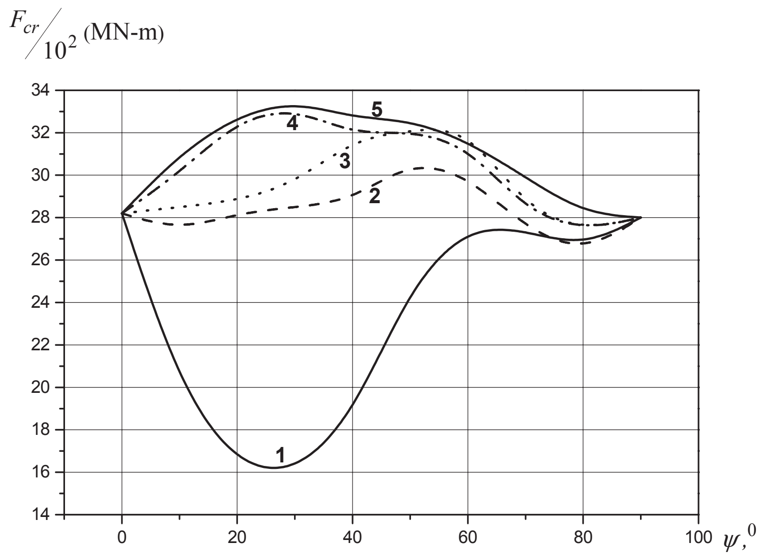

Fig. 2. Axial critical load distribution given shell material's winding angle of main direction of elasticity

Moreover, critical values being obtained in calculations without taking anisotropic material constants into account converge to their values faster for the shells with odd number of layers than for the shells with even number of cross-wound layers. This corresponds to the results of the calculations presented by Bespalova [1999], Semenyuk and Trach [2006], Trach [2008] for the values of free vibration frequencies of cylindrical shells. Figure 2 also shows that for the angles $\psi_i = \pm 70^0 \div 80^0$ an increase in the number of cross-wound layers does not have significant impact on the character of changes in the critical values of the axial load, as it does for smaller winding angles.

At the angles close to 0, the values of critical stresses slightly differ from each other for the shells with the number of layers bigger than one. According to the authors, it is connected to the reduction of influence of the constants a_{i6} $i = \overline{1,3}$ and which characterize the anisotropy of material under consideration. It is confirmed by the analysis of the dependences (5). Due to this, one can state that at angles ψ close to 0^o and 90^o these coefficients slightly affect the sizes of critical stresses. Thus, it is possible to claim that at such angles the shell is orthotropic.

CONCLUSIONS

As shown, the application of the recommended approach, based on the application of Bubnov-Galerkin procedure (with taking boundary conditions on surfaces and end edges of a cylindrical shells into account) as well as the numerical method of discrete orthogonalization, allows to solve three-dimensional problems of stability of cylindrical shells being under axial load, made of materials with one plane of elastic symmetry in a wide range of variety of geometric and mechanical characteristics.

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STATECZNOŚĆ OSIOWO ŚCISKANYCH TRÓJWYMIAROWYCH ANIZOTROPOWYCH POWŁOK CYLINDRYCZNYCH

Streszczenie. Przedstawiono próbę do rozwiązania problemu stateczności anizotropowych powłok cylindrycznych pod naciskiem osiowym, które opiera się na procedurze Bubnova-Galerkina przy wykorzystaniu warunków brzegowych na powierzchni i na krawędzi powłok cylindrycznych oraz na metodzie numerycznej ortogonalizacji dyskretnej. Rozwiązano problem stateczności powłok cylindrycznych wykonanych z materiału charakteryzującego się jedną płaszczyzną symetrii. Badano zależność obciążenia krytycznego od kąta obrotu głównych kierunków sprężystości. Wyniki obliczeń przedstawiono na wykresach i tabelach, ponadto przeprowadzono ich analizę.

Słowa kluczowe: stateczność, powłoki cylindryczne anizotropowe, metoda Bubnowa-Galorkina, trójwymiarowy stan

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