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ON THE EXISTENCE OF THE SINUSOIDAL-TYPE TEMPERATURE FLUCTUATIONS WHICH ARE INDEPENDENTLY SUPPRESSED BY THE PERIODIC TWO-PHASED CONDUCTING LAYER

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Abstract. The paper describes how to analytically solve hyperbolic system of partial differential equations describing the boundary effect phenomenon, obtained in the framework of the refined tolerance model of heat conduction in periodic composites [Kula and Wierzbicki 2015]. Basic unknowns of this model are coefficients of Fourier expansion representing the temperature field which represents an alternative exact description of the heat transfer in composite rigid conductors. The aim of this paper is to indicate two sinusoidal-type boundary impulses which cannot be transported independently by the two-phased periodic laminated composite. Investigation of the form of the suppression of thermal boundary fluctuations is important to allow the construction to minimize the destruction of composites caused by the rotations of fluctuation amplitudes which are transported by composite not independently.

Key words: heat transfer eguation, boundary effect behavior, Fourier expansion

FORMULATION OF THE PROBLEM

The paper deals with stationary heat transfer equation:

$$\mathrm{div}(K\nabla\theta) = -f \tag{1}$$

in which the temperature field $\theta = \theta(z,t)$, $z = (z^1, z^2, z^3) \in R^3$, is represented by the Fourier expansion (with respect to the basis $\varphi^0 \equiv u, \varphi^p$, p = 1, 2, ...):

$$\theta(z) = u(z) + \lambda a_p(x)\varphi^p(\xi) \tag{2}$$

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in the regular points $z \in \Omega \setminus \Gamma$ of the layer:

$$\Omega = \Phi \times (0, \delta) \subset R^3 \tag{3}$$

occupied by the composite in the reference configuration, cf. Figure 1.

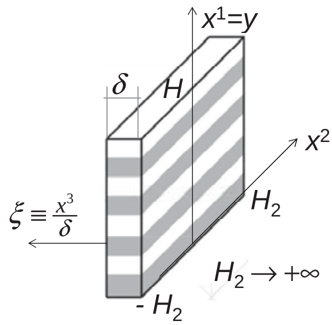

Fig. 1. The considered periodic layer

In equation (2) $z \equiv (\xi, x) \in \Phi \times (0, \delta)$ for $x \in \Phi = (0, H^2) \times (0, \delta) \subset R^2$ and $\xi = z^1 \in (0, H)$. Symbol Γ denotes here the planes separated homogeneous laminas. Since the considerations are restricted to the two-phased laminated periodic conductor. Heat conductivity constant $k = k(\xi)$ are assumed to be λ – periodic with respect to the ξ – variable. The microstructure parameter λ is equal to the sum $\lambda = \lambda^I + \lambda^{II}$ of lamina thickness. We also assume that $k(\xi) = k^I$, and $k(\xi) = k^{II}$, for $\xi \in (n\lambda, \lambda^I + n\lambda)$ and $\xi \in (\lambda^I + n\lambda, (n + 1)\lambda)$, $n = ..., -2, -1, 0, 1, 2, ...,$ respectively. In equation (1), $\mathrm{div} p \equiv \partial p_1 / \partial z^1 + \partial p_2 / \partial z^2 + \partial p_3 / \partial z^3$ and $\nabla \equiv [\partial / \partial z^1, \partial / \partial z^2, \partial / \partial z^3]^T$ for $p \equiv [p_1, p_2, p_3]^T$ and f represents heat source. Expansion equation (2) is here interpreted as in the *tolerance heat transfer extended model*, cf. equation (1), and hence there exists constant effective conductivity $k^{eff} > 0$ and constant coefficients $[k]^p = [[k]_1^p, [k]_2^p, [k]_3^p] \in R^3$ for which model equations:

$$\begin{aligned} &\nabla^T [K^{eff} \nabla u] = -\langle f \rangle \\ &\lambda^2 \nabla_\Phi{}^T \langle \varphi^p c \varphi^q \rangle \nabla_x a_q \} - \lambda(\langle \nabla_y{}^T \varphi^p K \varphi^q \rangle - \langle \nabla_y{}^T \varphi^q K \varphi^p \rangle) \nabla_x a_q + \\ &-\langle \nabla_y{}^T \varphi^q K \nabla_y \varphi^p \rangle a_p - [K]^p \nabla u = \lambda \langle \varphi^p f \rangle \end{aligned} \tag{4}$$

are satisfied. In equation (4) averaging operator:

$$\langle g \rangle (y) = \frac{1}{|\Delta|} \int_{y+\Delta} g(\zeta) d\zeta \tag{5}$$

for $\Delta \equiv [0, \lambda]$ and additional denotations $\nabla_\xi \equiv [\partial / \partial z^1, 0, 0]^T$, $\nabla_x \equiv [0, \partial / \partial z^2, \partial / \partial z^3]^T$ have been used. Under denotations:

(6)

$$A^{pq} = \langle \varphi^p k \varphi^q \rangle$$
$$B^{pq} = \langle \nabla \xi^T \varphi^p K \varphi^q \rangle - \langle \nabla_\xi{}^T \varphi^q K \varphi^p \rangle$$
$$C^{pq} = \langle \nabla_\xi{}^T \varphi^p K \nabla_\xi \varphi^q \rangle$$

the second equation from (4) will be rewritten in the form:

$$\lambda^2 \nabla_x{}^T (A^{pq} \nabla_x a_q) - \lambda B^{pq} \nabla_x a_q + C^{pq} a_q \left\langle [K]^p \nabla u \right\rangle = \lambda \langle \varphi^p f \rangle \tag{7}$$

Onto the temperature field the boundary conditions and the restriction $f \equiv 0$ will be imposed. For the averaged temperature field homogeneous boundary condition will be assumed:

$$u\,|_{(\xi,x)\in\partial\Omega} = 0 \tag{8}$$

and for the Fourier amplitudes constant boundary values:

$$a_p(\xi)\,|_{z^3=0} = a_p^0 = \text{const}, \quad a_p(\xi)\,|_{z^3=\delta} = a_p^\delta = \text{const} \tag{9}$$

It should be emphasized that equation (8) leads to the conclusion that $u \equiv 0$ almost everywhere in Ω and that Fourier amplitudes a_p should satisfy infinite system of equations:

$$\lambda^2 \nabla_x{}^T (A^{pq} \nabla_x a_q) - \lambda B^{pq} \nabla_x a_q + C^{pq} a_q = 0 \tag{10}$$

which will be referred to as *the boundary effect equations.*

Under the assumption that $L \to +\infty$ (10) reduces to the ordinary differential equation, since in this case $\partial a_p / \partial z^2 = 0$ can be assumed for any p and hence $\nabla_x = [0, \partial/\partial z^2, \partial/\partial z^3]$ reduces to $\nabla_x = [0, 0, \partial/\partial z^3]$. That is why in the subsequent consideration x will be treated as z^3 and ∇_x will be replaced by ∂_x.

FOURIER BASIS

For two-phased one-directionally periodic composite as a Fourier basis $\varphi^p(\xi)$ used in the temperature representation equation (2) will be taken into account the simplest orthogonal basis includes the following basis fluctuations:

$$g^p(\xi) = \begin{cases} \frac{1}{2} - \alpha_p [1 + \cos 2\pi p (\frac{\xi}{\lambda \eta^I} + 1)] & \text{dla} \quad -\lambda \eta^I \le \xi \le 0 \\ \frac{1}{2} - \alpha_p [1 + \cos 2\pi p (\frac{\xi}{\lambda \eta^I} + 1)]\,|_{y=0} & \text{dla} \quad 0 \le \xi \le \lambda \eta^{II} \end{cases}$$

$$g^{p+m}(\xi)=\begin{cases}\frac{1}{2}-\alpha_{p+m}[1+\cos 2\pi p(\frac{\xi}{\lambda\eta^{II}}-1)]|_{\xi=0} & \text{dla} \quad -\lambda\eta^{I}\leq\xi\leq 0\\ \frac{1}{2}-\alpha_{p+m}[1+\cos 2\pi p(\frac{\xi}{\lambda\eta^{II}}-1)] & \text{dla} \quad 0\leq\xi\leq\lambda\eta^{II}\end{cases}$$

$$g^{p+2m}(\xi)=\begin{cases}-\frac{1}{2}\cos(2p-1)\pi(\frac{\xi}{\lambda\eta^{I}}+1) & \text{dla} \quad -\lambda\eta^{I}\leq\xi\leq 0\\ -\frac{1}{2}\cos(2p-1)\pi(\frac{\xi}{\lambda\eta^{II}}-1) & \text{dla} \quad 0\leq\xi\leq\lambda\eta^{II}\end{cases} \tag{11}$$

where: $\lambda^{I}=\eta_{I}^{\lambda}$, $\lambda^{II}=\eta_{II}^{\lambda}$ and

$$\alpha_{p}=\frac{\lambda}{2}\bar{\alpha}\equiv\frac{\lambda}{2(2\eta_{I}+\eta_{II})}$$

$$\alpha_{p+m}=\frac{\lambda}{2}\underline{\alpha}\equiv\frac{\lambda}{2(\eta_{I}+2\eta_{II})} \tag{12}$$

lead to condition $\langle g^{p}\rangle=\langle g^{p+m}\rangle=\langle g^{p+2m}\rangle=0.$ Fluctuations equations (11) are divided into infinite number of three-elements packages indexed by the positive integer number $p = 1, ..., m$. Positive integer m numbered partial sums of the Fourier expansion equation (2) and the realization of the limit passage $m\to+\infty$ complete the infinite expansion equation (2). Unfortunately the family of fluctuations equations (11) is not orthogonal. The orthogonality procedure will be realized by:

$$\varphi^{p}=g^{p}+\alpha g^{p+m}$$
$$\varphi^{p+m}=g^{p}-\alpha g^{p+m} \tag{13}$$
$$\varphi^{p+2m}=g^{p+2m}$$

for

$$\alpha=\pm\frac{\langle kg^{p}g^{p}\rangle}{\langle kg^{p+m}g^{p+m}\rangle} \tag{14}$$

in which summation over repeated indices does not apply.

COEFFICIENTS CALCULATION

Bearing in mind equations (11) and (13) we can calculate coefficients (6). Firstly we obtain:

$$A^{pq}=\bar{\kappa}\delta^{pq},\quad A^{p+m\ q+m}=\underline{\kappa}\delta^{pq},\quad A^{p+2m\ q+2m}=\tilde{\kappa}\delta^{pq} \tag{15}$$

for the following values of constants $\bar{\kappa}$, $\underline{\kappa}$:

$$\bar{\kappa} = \frac{1}{4}\{k^I[(1-\frac{1}{2\eta_I+\eta_{II}})^2+\frac{1}{2(2\eta_I+\eta_{II})^2}+\alpha^2(1-\frac{1}{\eta_I+2\eta_{II}})^2]\eta_I +$$
$$+k^{II}[(1-\frac{1}{2\eta_I+\eta_{II}})^2+\alpha^2[(1-\frac{1}{\eta_I+2\eta_{II}})^2+\frac{1}{2(\eta_I+2\eta_{II})^2}]\eta_{II}\}$$

$$\underline{\kappa} = \frac{1}{4}\{k^I[(1-\frac{1}{2\eta_I+\eta_{II}})^2+\frac{1}{2(2\eta_I+\eta_{II})^2}+\alpha^2(1-\frac{1}{\eta_I+2\eta_{II}})^2]\eta_I +$$
$$+k^{II}[(1-\frac{1}{2\eta_I+\eta_{II}})^2+\alpha^2[(1-\frac{1}{\eta_I+2\eta_{II}})^2+\frac{1}{2(\eta_I+2\eta_{II})^2}]\eta_{II}\} \tag{16}$$

and $\underset{\sim}{\kappa}$

$$\tilde{\kappa} = \frac{1}{8}\langle k\rangle \tag{17}$$

Similarly $C^{pq} = \bar{\gamma}p^2$, $C^{p+m\,q+m} = \underline{\gamma}p^2$, $C^{p+2m\,q+2m} = \underset{\sim}{\gamma}(2p-1)^2$ while $p = q$ and $C^{pq} = C^{p+m\,q+m} = C^{p+2m\,q+2m} = 0$ otherwise, for

$$\bar{\gamma} = \underline{\gamma} = 2\pi^2\{\frac{k_I}{\eta_I}[(1-\frac{1}{2\eta_I+\eta_{II}})^2+\frac{1}{2(2\eta_I+\eta_{II})^2}+\alpha^2(1-\frac{1}{\eta_I+2\eta_{II}})^2]+$$
$$+\frac{k_{II}}{\eta_{II}}[(1-\frac{1}{2\eta_I+\eta_{II}})^2+\alpha^2[(1-\frac{1}{\eta_I+2\eta_{II}})^2+\frac{1}{2(\eta_I+2\eta_{II})^2}]\} \tag{18}$$

$$\tilde{\gamma} = \frac{1}{2}(2p-1)^2\pi^2(\frac{k_I}{\eta_I}+\frac{k_{II}}{\eta_{II}})$$

Moreover, for matrix B we conclude that its only not vanishing elements includes in sub-matrices with elements:

$$B^{p+2m\;q} = -B^{q\;p+2m} = (\eta_I k^I\bar{\alpha} - \alpha\eta_{II}k^{II}\underline{\alpha})\tilde{B}^{pq}$$
$$B^{p+2m\;q+m} = -B^{q+m\;p+2m} = (\eta_I k^I\bar{\alpha} + \alpha\eta_{II}k^{II}\underline{\alpha})\tilde{B}^{pq} \tag{19}$$

for $p, q = 1, ..., m$ and

$$\tilde{B}^{pq} = \frac{4p^2}{4p^2+(2q-1)^2} - \frac{4q^2}{4q^2+(2p-1)^2} \tag{20}$$

Matrix with elements given by equation (20) will be referred to as *suppression matrix*. It must be emphasized that, under (19) and (20), matrices B and $\tilde{B}$ are quadratic of the ranges $3m\times 3m$ and $m\times m$, respectively. Moreover, both are anti-symmetric and hence both have imaginary or vanishes eigenvalues.

FINAL SOLUTIONS

Now we are to investigate the solution to the formulated Dirichlet problem for a free single boundary fluctuation impulse formed by the basic impulses taken from the p-th package from equations (11). Hence it should be a certain impulse of the form $\theta = a_p\varphi^p + a_{p+m}\varphi^{p+m} + a_{p+2m}\varphi^{p+2m}$ in which coefficients a_p, a_{p+m}, a_{p+2m} are solutions to the mentioned Dirichlet problem in $\Omega\setminus\Gamma$ with constant boundary values given by special case of boundary conditions (9):

$$
\begin{aligned}
&a_p(x)|_{x=0} = a_p^0, \quad a_{p+m}(x)|_{x=0} = a_{p+m}^0, \quad a_{p+2m}(x)|_{x=0} = a_{p+2m}^0 \\
&a_p(x)|_{x=\delta} = a_p^\delta, \quad a_{p+m}(x)|_{x=0} = a_{p+m}^\delta, \quad a_{p+2m}(x)|_{x=0} = a_{p+2m}^\delta
\end{aligned}
\tag{21}
$$

for the mentioned values p, m and vanishing other boundary constant Fourier amplitudes used in equation (9). Hence p and m, $m \geq p$, should be fixed pair of positive integers. It easy to verify that all elements of the suppression matrix are in the mentioned case equal to zero and hence boundary effect equations, under the diagonality of matrices A, C and under boundary conditions (21), reduce to the three independent equations for φ^p, φ^{p+m}, φ^{p+2m} :

$$
\begin{aligned}
&\lambda^2\overline{k}\partial_3^2 a_p + \vec{\gamma}\, p^2 a_p = 0 \\
&\lambda^2\underset{\leftarrow}{k}\partial_3^2 a_{p+m} + \underset{\leftarrow}{\gamma}\, p^2 a_{p+m} = 0 \\
&\lambda^2\underset{\sim}{k}\partial_3^2 a_{p+2m} + \underset{\sim}{\gamma}(2p-1)^2 a_{p+2m} = 0
\end{aligned}
\tag{22}
$$

with solutions

$$
a_p(x^3) = D_p^0 \sinh\sinh\left(p\sqrt{\frac{\overline{\gamma}}{\overline{k}}}\frac{x^3-\delta}{\lambda}\right) + D_p^\delta \sinh\left(p\sqrt{\frac{\overline{\gamma}}{\overline{k}}}\frac{x^3}{\lambda}\right)
\tag{23}
$$

for the first

$$
a_{p+m}(x^3) = D_{p+m}^0 \sinh\left(p\sqrt{\frac{\underset{\leftarrow}{\gamma}}{\underset{\leftarrow}{k}}}\frac{x^3-\delta}{\lambda}\right) + D_{p+m}^\delta \sinh\left(p\sqrt{\frac{\underset{\leftarrow}{\gamma}}{\underset{\leftarrow}{k}}}\frac{x^3}{\lambda}\right)
\tag{24}
$$

for the second

$$
a_{p+2m}(x^3) = D_{p+2m}^0 \sinh\left[(2p-1)\sqrt{\frac{\underset{\sim}{\gamma}}{\underset{\sim}{k}}}\frac{x^3-\delta}{\lambda}\right] + D_{p+m}^\delta \sinh\left[(2p-1)\sqrt{\frac{\underset{\sim}{\gamma}}{\underset{\sim}{k}}}\frac{x^3}{\lambda}\right]
\tag{25}
$$

for the third.

Coefficients $D_p^-, D_p^+, D_{p+m}^-, D_{p+m}^+, D_{p+2m}^-, D_{p+2m}^+$ are determined by boundary conditions (21):

$$
\begin{aligned}
D_p^0 &= -\frac{a_{i+2m}^0}{\sinh\left(p\sqrt{\frac{\bar{\gamma}}{\bar{k}}}\frac{\delta}{\lambda}\right)}, \quad D_p^\delta = \frac{a_{i+2m}^\delta}{\sinh\left(p\sqrt{\frac{\bar{\gamma}}{\bar{k}}}\frac{\delta}{\lambda}\right)} \\
D_{p+m}^0 &= -\frac{a_{i+2m}^0}{\sinh\left(p\sqrt{\frac{\underset{\leftarrow}{\gamma}}{\underset{\leftarrow}{k}}}\frac{\delta}{\lambda}\right)}, \quad D_{p+m}^\delta = \frac{a_{i+2m}^\delta}{\sinh\left(p\sqrt{\frac{\underset{\leftarrow}{\gamma}}{\underset{\leftarrow}{k}}}\frac{\delta}{\lambda}\right)} \\
D_{p+2m}^0 &= -\frac{a_{i+2m}^0}{\sinh\left[(2p-1)\sqrt{\frac{\underset{\sim}{\gamma}}{\underset{\sim}{k}}}\frac{\delta}{\lambda}\right]}, \quad D_{p+2m}^\delta = \frac{a_{i+2m}^\delta}{\sinh\left[(2p-1)\sqrt{\frac{\underset{\sim}{\gamma}}{\underset{\sim}{k}}}\frac{\delta}{\lambda}\right]}
\end{aligned}
\tag{26}
$$

Formulas (23), (24) and (25) determine the exact solution to the mentioned Dirichlet problem and hence consist the partial but exact description of the boundary effect behavior. Physical character of coefficients $\sqrt{\frac{\bar{\gamma}}{\bar{k}}}\frac{p}{\lambda}$, $\sqrt{\frac{\underset{\leftarrow}{\gamma}}{\underset{\leftarrow}{k}}}\frac{p}{\lambda}$, $\frac{2p-1}{\lambda}\sqrt{\frac{\underset{\sim}{\gamma}}{\underset{\sim}{k}}}$ suggest that they are proper physical measurement of the *intensity of the boundary effect behavior* in the case of every boundary fluctuations mentioned by the boundary loadings (21). That is why nondimensional parameters $p\sqrt{\frac{\bar{\gamma}}{\bar{k}}}$, $p\sqrt{\frac{\underset{\leftarrow}{\gamma}}{\underset{\leftarrow}{k}}}$, $(2p-1)\sqrt{\frac{\underset{\sim}{\gamma}}{\underset{\sim}{k}}}$ will be referred to as intensities of the boundary effect behavior for every of loadings (21), respectively. It must be emphasized that obtained solutions not depend on the positive integer *m* determined the size of the partial sum of the used Fourier expansion (2).

CONCLUDING REMARKS

The paper is based on a certain modification of the tolerance modeling approach, proposed and used in many monographs and contributions by Czesław Woźniak, cf. for example: Woźniak and Wierzbicki [2000], Nagórko [2008], Woźniak ed. [2009, 2010], Jędrysiak [2010], Michalak [2010]. This modification lets to the possibility of investigated exact solutions to the heat transfer equation for periodic composites. These solutions are investigated as Fourier expansions. The applied method of modeling results in the model equations consist of the single equation for the averaged temperature field and infinite system of the second order hyperbolic partial differential equations for the Fourier amplitudes.

It should be emphasized that solutions (23), (24), (25) of the boundary effect equation usually depends on the number *m* determined the size of the partial sum of used Fourier expansion. In the paper it has been shown that fluctuations from the same package defined by equations (11) are transferring independently by the laminated two-phased periodic conductor. In the references monographs and article strictly connected with the paper have been listed.

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O ISTNIENIU SINUSOIDALNYCH FLUKTUACJI TERMICZNYCH TŁUMIONYCH NIEZALEŻNIE PRZEZ DWUFAZOWĄ WARSTWĘ KOMPOZYTOWĄ

Streszczenie. W artykule opisano, jak rozwiązać analitycznie hiperboliczny układ równań różniczkowych cząstkowych, opisujący zjawisko efektu brzegowego w ramach rozszerzonego tolerancyjnego modelu przewodnictwa ciepła w kompozytach periodycznych [Kula i Wierzbicki 2015]. Podstawowymi niewiadomymi tego modelu są współczynniki rozwinięcia pola temperatury w szereg Fouriera. Szereg ten jest dokładnym rozwiązaniem równania przewodnictwa ciepła. Celem niniejszego artykułu jest wykazanie, że trzy sinusoidalne impulsy brzegowe, pochodzące z tego samego pakietu fluktuacji wybranego spośród nieskończonej rodziny pakietów tworzących układ bazowy Fouriera, są transportowane niezależnie do wnętrza dwuskładnikowego periodycznego przewodnika laminowanego. Badanie tego typu zjawiska pozwala na odpowiedź na pytanie, jaka powinna być budowa geometryczna kompozytu, by nie dopuszczała do powstawania rotacji amplitud fluktuacji przenoszonych przez kompozyty zależnie od siebie. Rotacje te są prawdopodobnie najistotniejszą przyczyną niszczenia kompozytów podczas ich eksploatacji.

Słowa kluczowe: równanie ciepła zjawisko efektu brzegowego, szeregi Fouriera

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